Special Constructions of MAD families

Definition

1) $A, B \subseteq \omega$, A and B are almost disjoint $A \cap B$ is finite

2) $\mathcal{A} \subseteq [\omega]^\omega$ is an AD family if $\forall A, B \in \mathcal{A}$ if $A \neq B$ then A and B are almost disjoint

3) An infinite $\mathcal{A} \subseteq [\omega]^\omega$ AD family is MAD if it is maximal

Definitions $\mathcal{A}$ AD family

1) $\mathcal{I}(\mathcal{A})$ the ideal generated by $\mathcal{A}$

2) $\mathcal{I}(\mathcal{A})^+ = P(\omega) \setminus \mathcal{I}(\mathcal{A})$

3) $\mathcal{I}(\mathcal{A})^{++}$

$X \in \mathcal{I}(\mathcal{A})^{++} \iff$ there is $\{A_n \mid n \in \omega\} \subseteq \mathcal{A}$ such that $|X \cap A_n| = \omega$ for every $n \in \omega$

4) $\mathcal{A}^\perp = \{X \subseteq \omega \mid \forall A \in \mathcal{A} (X \cap A \text{ is finite})\}$

Prop Let $\mathcal{A}$ be an AD family. The following are equivalent:

1) $\mathcal{A}$ is MAD

2) $\mathcal{I}(\mathcal{A})^+ = \mathcal{I}(\mathcal{A})^{++}$

3) $\mathcal{A}^\perp = [\omega]^{<\omega}$

Def (Hechler) Let $\mathcal{A}$ be an MAD family. We say that $\mathcal{A}$ is *completely separable* if for every $X \in \mathcal{I}(\mathcal{A})^+$ $\exists A \in \mathcal{A}\ (A \subseteq X)$

$$\boxed{\text{Prop If } \mathcal{A} \text{ is completely separable and } X \in \mathcal{I}(\mathcal{A})^+, \text{ then } |\{A \in \mathcal{A} \mid A \subseteq X\}| = \mathfrak{c} \quad (\mathfrak{c} \text{ is the size of } |\mathbb{R}|)}$$

Hint: Prove there is an AD family $\mathcal{B} \subseteq P(X) \cap \mathcal{I}(\mathcal{A})^+$ of size $\mathfrak{c}$

problem (Erdös–Shelah)

Is there a completely separable MAD family?

Partial results:

1) (Hechler) Yes under MA

2) (Simon–Balcar) Yes under several hypothesis:

$\mathfrak{b}=\mathfrak{d}$, $\mathfrak{c}\leq\mathfrak{a}$, $\mathfrak{s}=\omega_1$

3) (Simon) Yes if we do not require maximality (change $\mathcal{I}(\mathcal{A})^{+}$ for $\mathcal{I}(\mathcal{A})^{++}$)

4) Shelah Yes, "almost always"

a) Yes if $\mathfrak{s} < \mathfrak{a}$

b) Yes if $\mathfrak{s} = \mathfrak{a}$ + an innocent hypothesis

c) Yes if $\mathfrak{s} = \mathfrak{a}$ + an innocent hypothesis

5) Mildenberger, Raghavan, Steprāns

Yes if $\mathfrak{s} = \mathfrak{a}$

(they eliminated the extra hypothesis used by Shelah and provided an uniform proof for $\mathfrak{s} \leq \mathfrak{a}$, which is the proof we will see in this tutorial).

(Shelah)

a) Yes if $\mathfrak{s} < \mathfrak{a}$

b) Yes if $\mathfrak{s} = \mathfrak{a}$ + ~~PCF hypothesis~~

c) Yes if $\mathfrak{a} < \mathfrak{s}$ + PCF hypothesis

Def $\mathfrak{a}$ almost disjointness number is the smallest size of MAD family

$$\omega \leq \mathfrak{a} \leq \mathfrak{c}$$

Lemma Let $\mathcal{A}$ be an AD family and $\{X_n \mid n\in\omega\} \subseteq \mathcal{I}(\mathcal{A})^+$ decreasing sequence. There is $Y \in \mathcal{I}(\mathcal{A})^+$ such that $Y \subseteq^* X_n$ for every $n\in\omega$ (i.e. Y is a pseudointersection of $\{X_n \mid n\in\omega\}$)

In case $\mathcal{X}=\{X_n \mid n\in\omega\}$ has a pseudointersection in $\mathcal{A}^+$, we take that one and we are done. We assume that there are no pseudointersection in $\mathcal{A}^+$. In this way, we can find $\{B_n \mid n\in\omega\}$ and $\{A_n \mid n\in\omega\}$ such that:

1) $B_n \in \mathcal{I}(\mathcal{A})$, $A_n \in \mathcal{A}$.

2) $B_n \subseteq A_n$.

3) If $n \neq m \Rightarrow A_n \neq A_m$.

4) B_n is a pseudointersection of X.

We can now find $C \in [\omega]^\omega$ a pseudointersection of X such that $|C \cap B_n| = \omega$ for every $n \in \omega$, so $C \in \mathcal{I}(\mathcal{A})^+$.

$\square$

Prop If $\mathfrak{a} = \mathfrak{c} \Rightarrow$
There is a completely separable MAD family

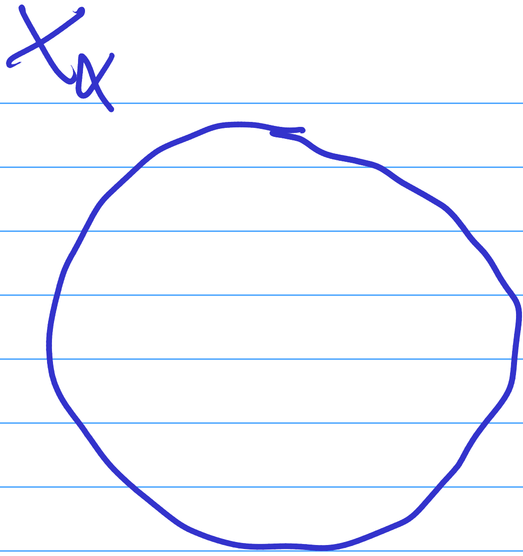
Let $[\omega]^{\omega} = \{X_\alpha \mid \alpha < \mathfrak{c}\}$ we will recursively construct $\mathcal{A} = \{A_\alpha \mid \alpha < \mathfrak{c}\}$ such that:

1) $\mathcal{A}_{<\alpha} = \{A_\xi \mid \xi < \alpha\}$ is AD
2) If $X_\alpha \in \mathcal{I}(\mathcal{A}_{<\alpha})^+$, then

$A_\alpha \subseteq X_\alpha$.

We are at step α

Assume $X_\alpha \in \mathcal{I}(\mathcal{A}_{<\alpha})^+$



Let $\mathcal{B} = \{ A_\xi \cap X_\alpha \mid \xi < \alpha \} \setminus [\omega]^{<\omega}$

$\mathcal{B}$ is an AD family of size less than $\mathfrak{a} = \mathfrak{c}$

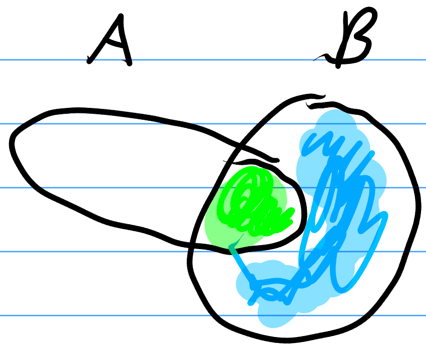
$\Rightarrow$ there is $A_\alpha \subseteq X_\alpha$

$A_\alpha \in \mathcal{B}^\perp \Rightarrow A_\alpha \in \mathcal{A}_{<\alpha}^\perp$.

$\square$

Def $A, B \subseteq \omega$

1) A **splits** B if

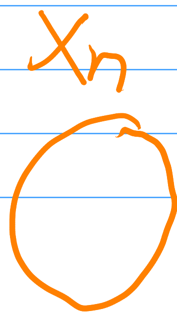


$A \cap B$, $B \setminus A$ are infinite

2) $\mathcal{S} \subseteq [\omega]^\omega$ is a **splitting family** if $\forall B \in [\omega]^\omega$ there $S \in \mathcal{S}$ S splits B.

3) $\mathfrak{s}$ **(the splitting number)** is the smallest size of a splitting family

$\omega < \mathfrak{s} \leq \mathfrak{c}$

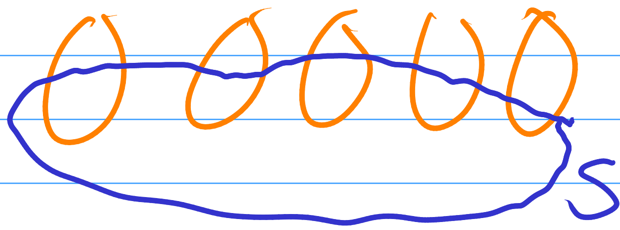


Lemma $\mathfrak{s}$ has uncountable cofinality

Theorem (Dow-Shelah) It is consistent that $\mathfrak{s}$ is singular

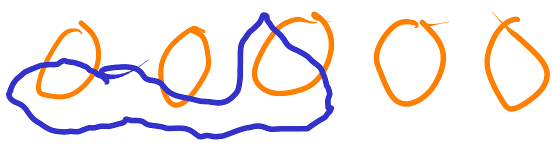
Def Let $\bar{X} = \{X_n \mid n \in \omega\} \subseteq [\omega]^\omega$ and $S \in [\omega]^\omega$

a) S ω-split $\bar{X}$ if $\forall n \in \omega$ S splits X_n

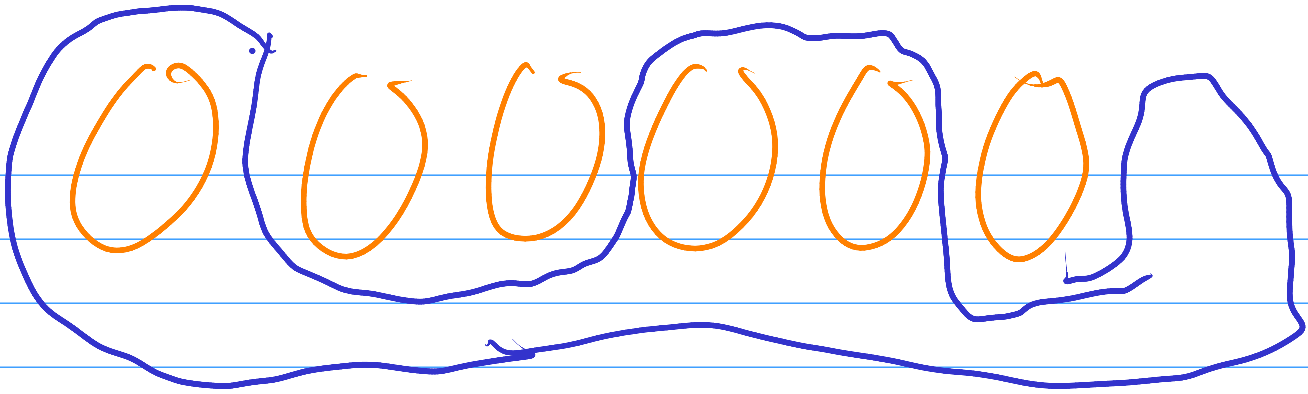


b) S (ω,ω)-splits $\bar{X}$ if the sets

$$\left.\begin{array}{l}\{n \mid |S \cap X_n| = \omega\} \\ \{n \mid |X_n \setminus S| = \omega\}\end{array}\right\} \text{ are infinite}$$



(example)



c) $\mathcal{S} \subseteq [\omega]^\omega$ is a (ω,ω)-splitting family if for every $\bar{X} = \{X_n \mid n \in \omega\} \subseteq [\omega]^\omega$ there is $s \in \mathcal{S}$ such that s (ω,ω)-splits $\bar{X}$

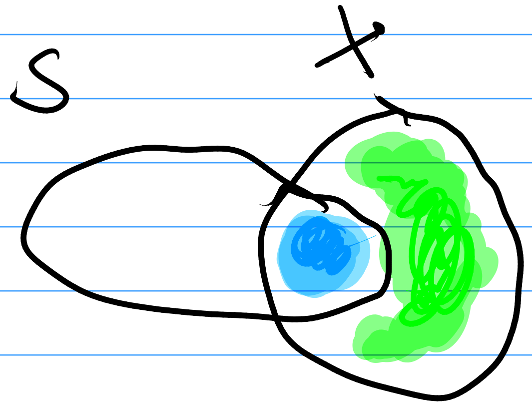
If $\mathcal{S}$ is a (ω,ω)-splitting family $\Rightarrow$ $\mathcal{S}$ is a splitting family

Prop (Mildenberger, Raghavan, Steprāns)

There is a (ω,ω)-splitting family of size $\mathfrak{s}$

Def Let $\mathcal{S}$ be a family and $\mathcal{I}$ an ideal. We say that $\mathcal{S}$ is an **$\mathcal{I}$-splitting family**

$\forall X \in \mathcal{I}^+$ there is $S \in \mathcal{S}$ such that $S \cap X, X \setminus S \in \mathcal{I}^+$



(where $\mathcal{I}^+ = \mathcal{P}(\omega) / \mathcal{I}$)

Prop

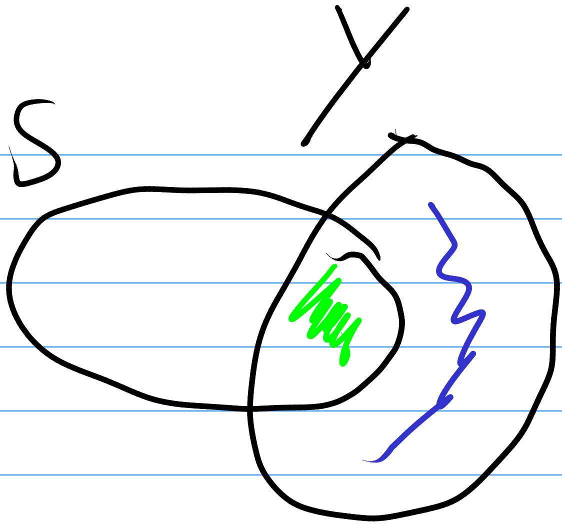
Let $\mathcal{S}$ be an (ω,ω)-splitting family. If $\mathcal{A}$ is an AD family $\Rightarrow$ $\mathcal{S}$ is $\mathcal{I}(\mathcal{A})$-splitting

Let $\mathcal{S}$ be (ω,ω)-splitting family and $\mathcal{A}$ an AD family. Let $X \in \mathcal{I}(\mathcal{A})^+$

case $X \notin \mathcal{I}(\mathcal{A})^{++}$

there is $Y \in [X]^\omega$ such that $Y \in \mathcal{A}^\perp$ (Y is almost disjoint with every element of $\mathcal{A}$)

Since $\mathcal{S}$ is (ω,ω)-splitting, in particular, it is splitting, so there is $S \in \mathcal{S}$ such that



$S \cap Y$, $Y \setminus S$ are infinite

Since $Y \in \mathcal{A}^{\perp}$ every infinite subset of Y is $\mathcal{I}(\mathcal{A})^{+}$

$\Rightarrow S \cap Y, Y \setminus S \in \mathcal{I}(\mathcal{A})^{+}$ ✓

Case $X \in \mathcal{I}(\mathcal{A})^{++}$

So, there is $\{A_n \mid n \in \omega\} \subseteq \mathcal{A}$ such that $|A_n \cap X| = \omega$ for every $n \in \omega$

Let $Y_n = A_n \cap X$

We look at $\{Y_n \mid n \in \omega\}$

Since $\mathcal{S}$ is (ω,ω)-splitting, there is $S \in \mathcal{S}$ such that:

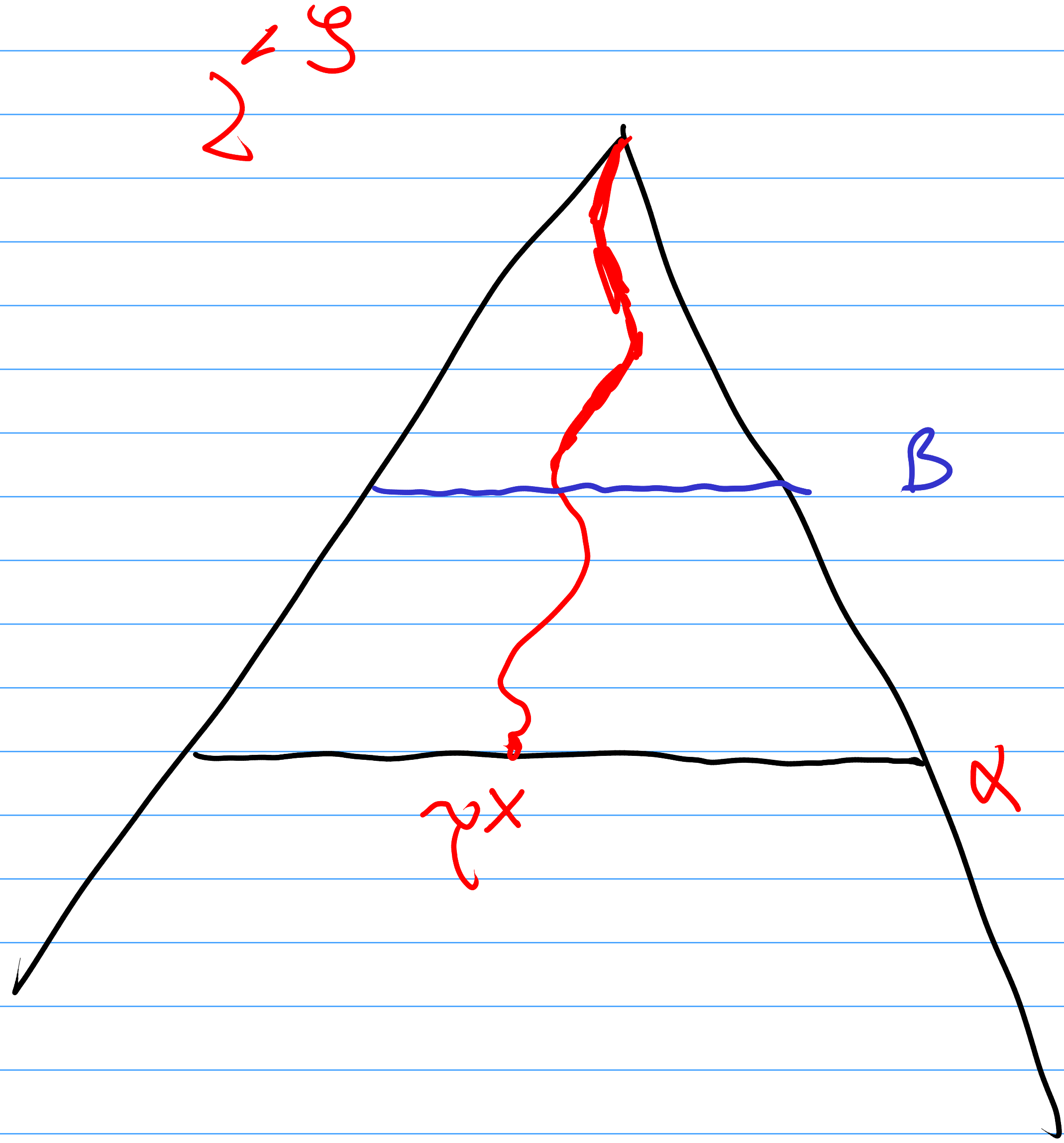
$$\left.\begin{array}{l}\{n \mid |Y_n \cap S| = \omega\} \\ \{n \mid |Y_n \setminus S| = \omega\}\end{array}\right\} \text{ are infinite}$$

Now, note that $S \cap X$ has infinite intersection with infinitely many elements of A

$S \setminus X$ has infinite intersection with infinitely many elements of A

$\Rightarrow S \cap X,\ S \setminus X \in \mathcal{I}(A)^+$ ✓

$\square$



Fix $\mathfrak{s} = \{S_\alpha \mid \alpha \in \mathfrak{s}\}$ (ω,ω)-splitting family

Notation: If $A \subseteq \omega$

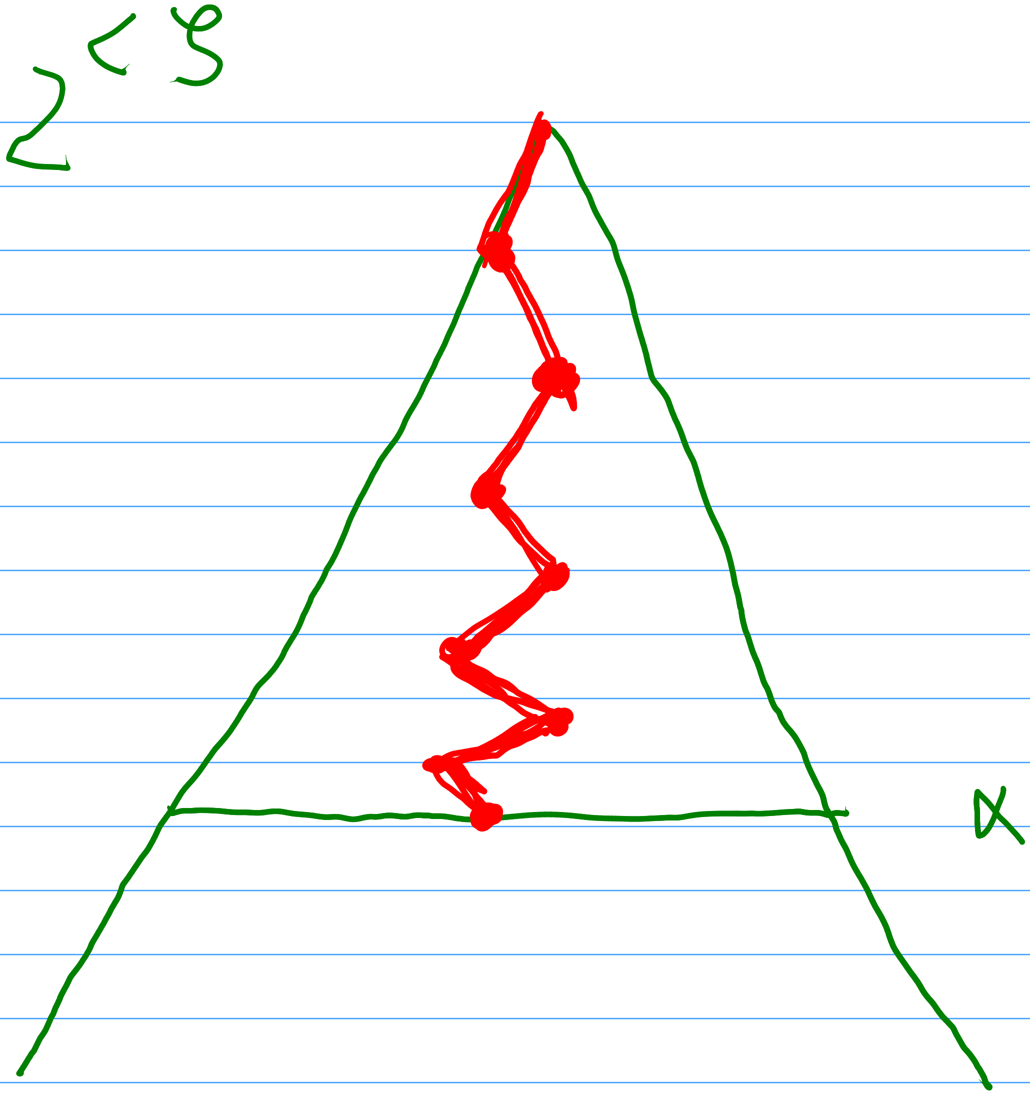
$A^0 = A$, $A^1 = \omega \setminus A$

lemma

Let $\mathcal{A}$ be an AD family and $X \in \mathcal{I}^+(\mathcal{A})$. There are $\alpha < \mathfrak{s}$ and $\tau_X^{\mathcal{A}} \in 2^\alpha$ such that:

1) If $\beta < \alpha \Rightarrow X \cap S_\beta^{1-\tau_X^{\mathcal{A}}(\beta)} \in \mathcal{I}(\mathcal{A})$

2) $X \cap S_\alpha, X \setminus S_\alpha \in \mathcal{I}(\mathcal{A})^+$



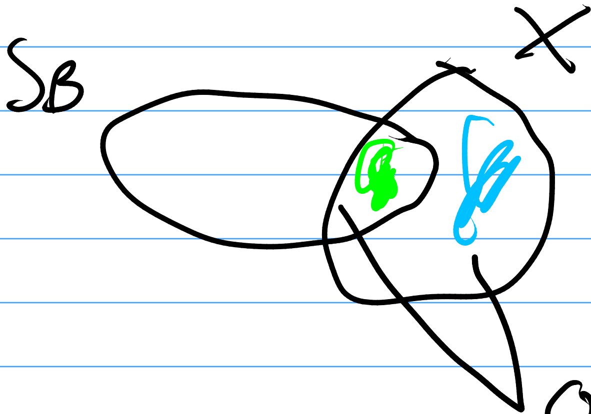
$X \in \mathcal{I}(\mathcal{A})^+$. We know that there is $S_\alpha \in \mathfrak{S}$ such that $X \cap S_\alpha, X \setminus S_\alpha \in \mathcal{I}(\mathcal{A})^+$.

Take the first α

So, if $\beta < \alpha$ then

either $X \cap S_\beta \subseteq \Upsilon(A)$ or

$X \setminus S_\beta \subseteq \Upsilon(A)$



One of this is small

"Most" of X is contained in S_β or in $W \setminus S_\beta$

Note that if $Y \subseteq X$, $Y \in \Upsilon(A)^+$, then

$$\tau_X^A \subseteq \tau_Y^A$$

Theorem (Shelah + Mildenberger, Raghavan, Steprāns)

If $\mathfrak{s} \leq \mathfrak{a}$, then there is a completely separable MAD family

Let $[\omega]^\omega = \{X_\alpha \mid \alpha < \mathfrak{c}\}$. We recursively define

$\mathcal{A} = \{A_\alpha \mid \alpha < \mathfrak{c}\}$

$\{\sigma_\alpha \mid \alpha < \mathfrak{c}\} \subseteq 2^{<\mathfrak{s}}$

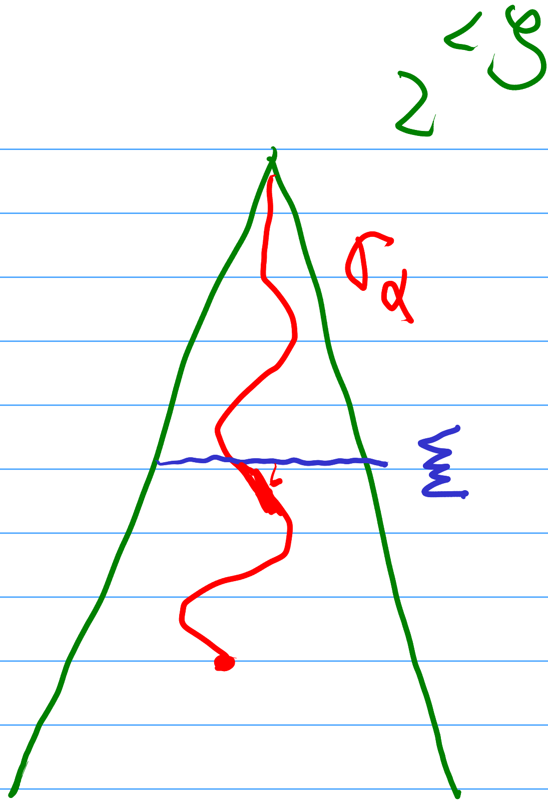
1) $\mathcal{A}_{<\alpha} = \{A_\xi \mid \xi < \alpha\}$ is AD

2) If $X_\alpha \in \mathcal{I}(\mathcal{A}_{<\alpha})^+$, then $A_\alpha \subseteq X_\alpha$

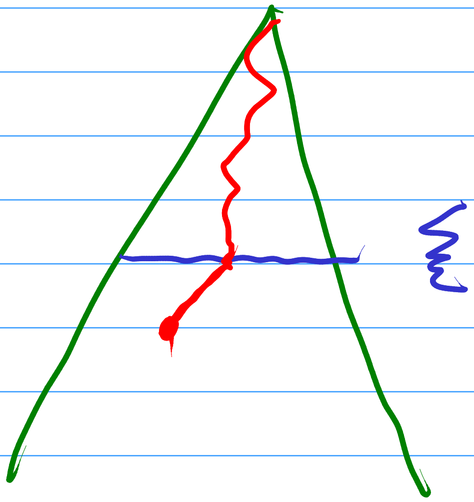
3) $\alpha \neq \beta \Rightarrow \sigma_\alpha \neq \sigma_\beta$

4) If $\xi < \mathrm{dom}(\sigma_\alpha)$

$A_\alpha \subseteq^* S_\xi^{\sigma_\alpha(\xi)}$



$\sigma_\alpha(\xi) = 0$



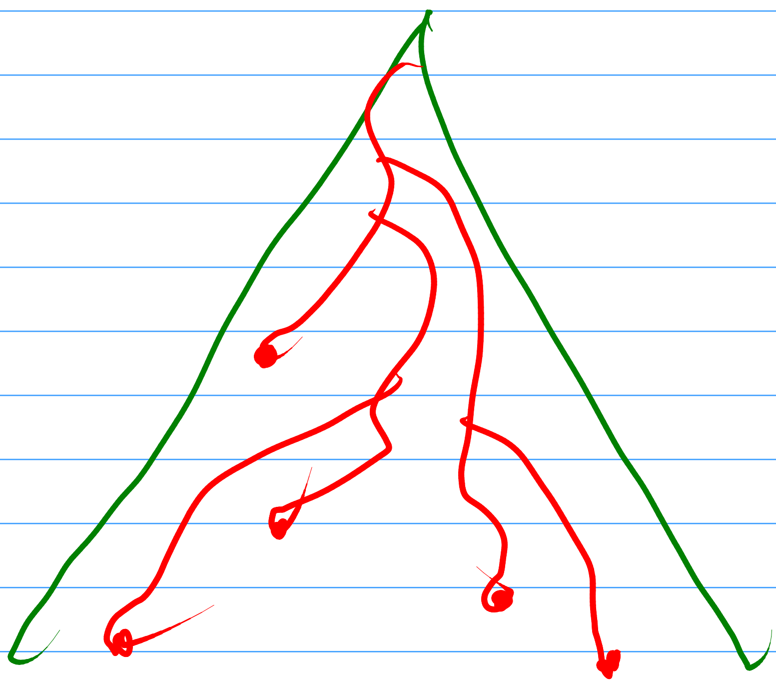
$\Rightarrow A_\alpha \subseteq^* S_\xi$

$\sigma_\alpha(\xi) = 1$

$\Rightarrow A_\alpha \subseteq^* \omega \setminus S_\xi$

Assume we are at step δ.

If $X_\delta \in \mathcal{I}(\mathcal{A}_{<\delta})^+$ let $X = X_\delta$ otherwise let X be any $\mathcal{I}(\mathcal{A}_{<\delta})^+$-set



Now, we will recursively find

$\{X_s \mid s \in 2^{<\omega}\} \subseteq \mathcal{I}(A_{<\delta})^+$,

$\{\eta_s \mid s \in 2^{<\omega}\} \subseteq 2^{<\xi}$

$\{\alpha_s \mid s \in 2^{<\omega}\} \subseteq \xi$ such that:

1) $X_\emptyset = X$

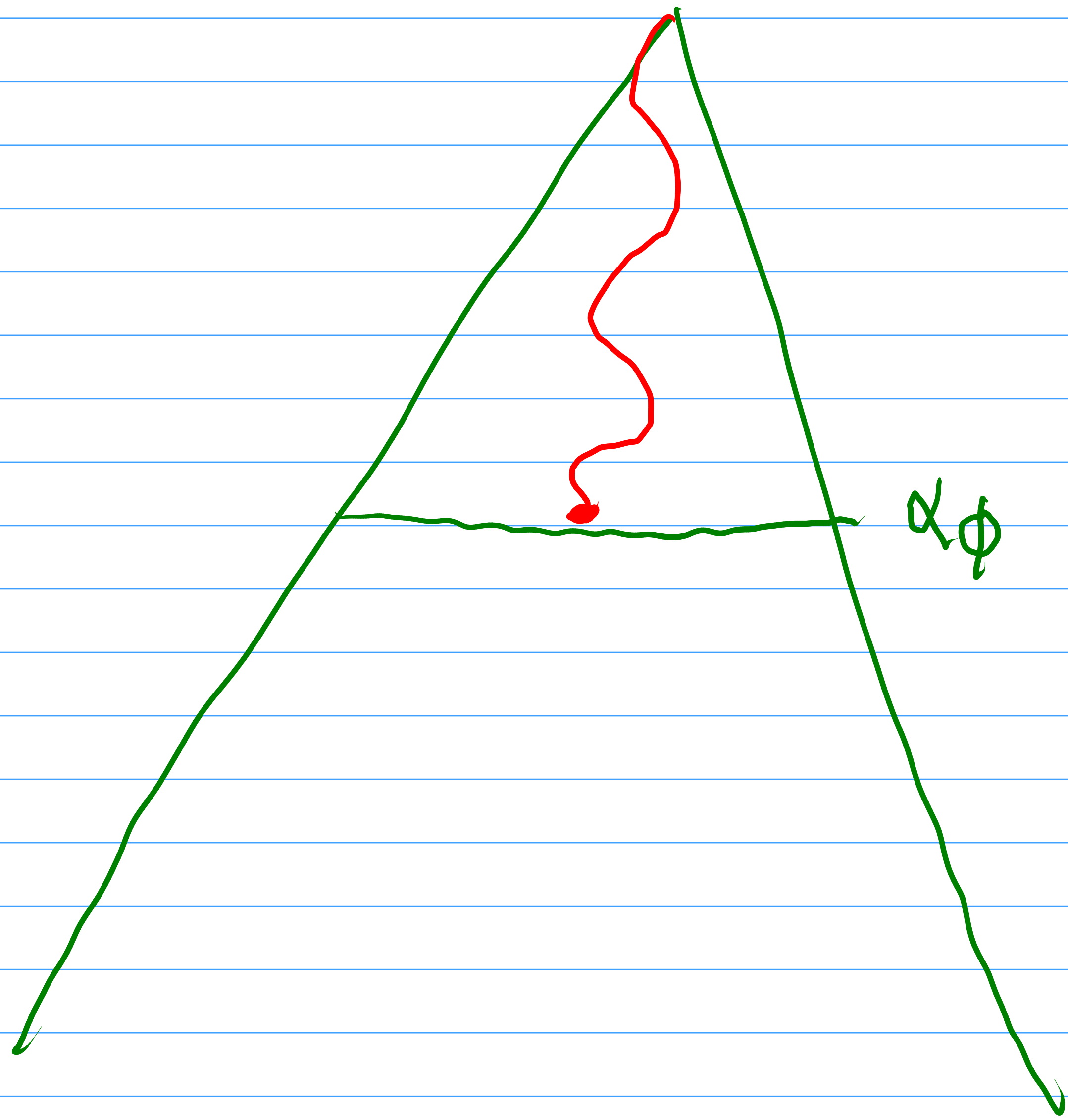
2) $\eta_s = \tau^{A_\delta}_{X_s}$

3) $\alpha_s = \mathrm{dom}(\eta_s)$

3) $X_{s^\frown 0} = X_s \cap S_{\alpha_s}$

$X_{s^\frown 1} = X_s \setminus S_{\alpha_s}$

Start with X



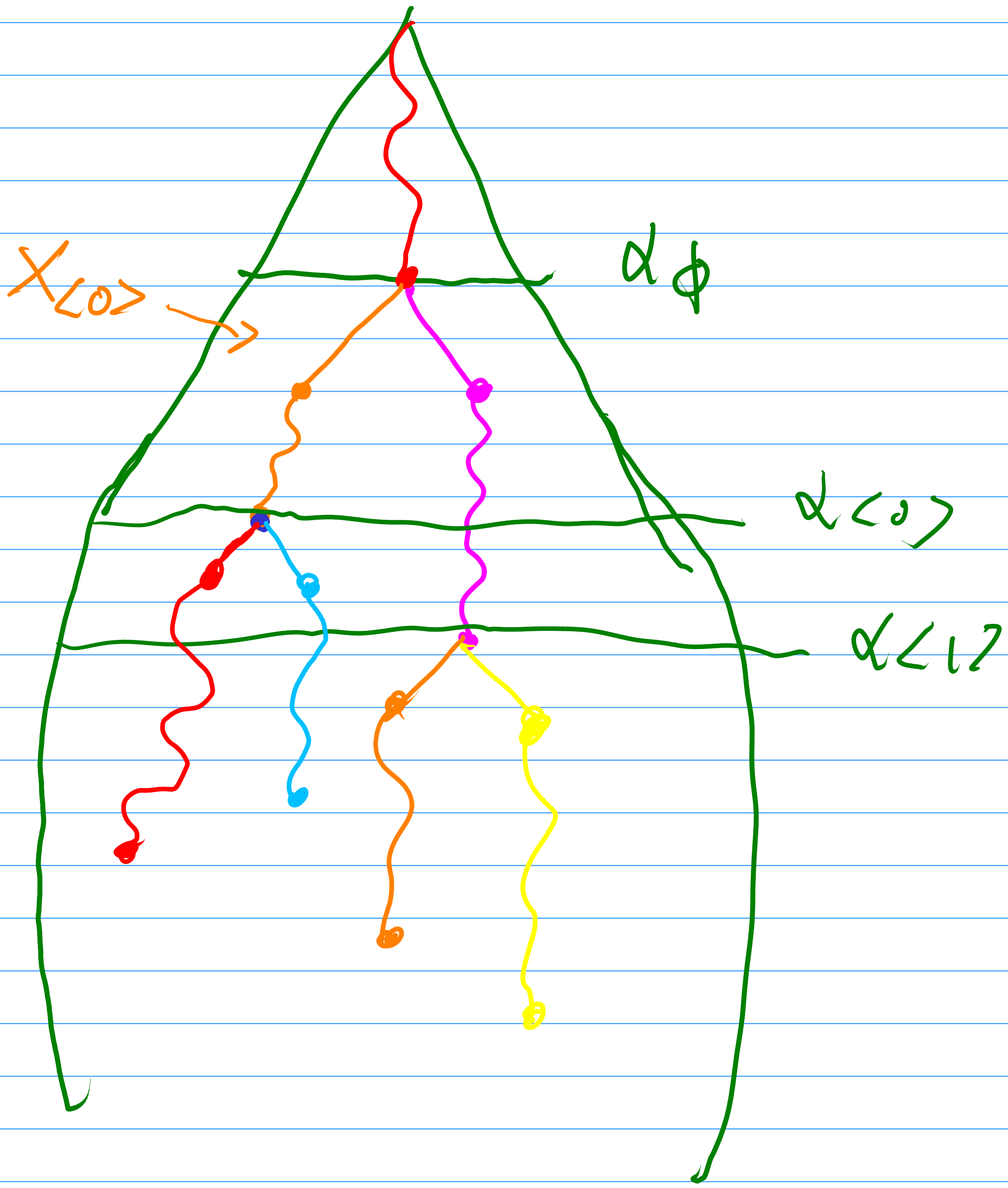
We get path as before

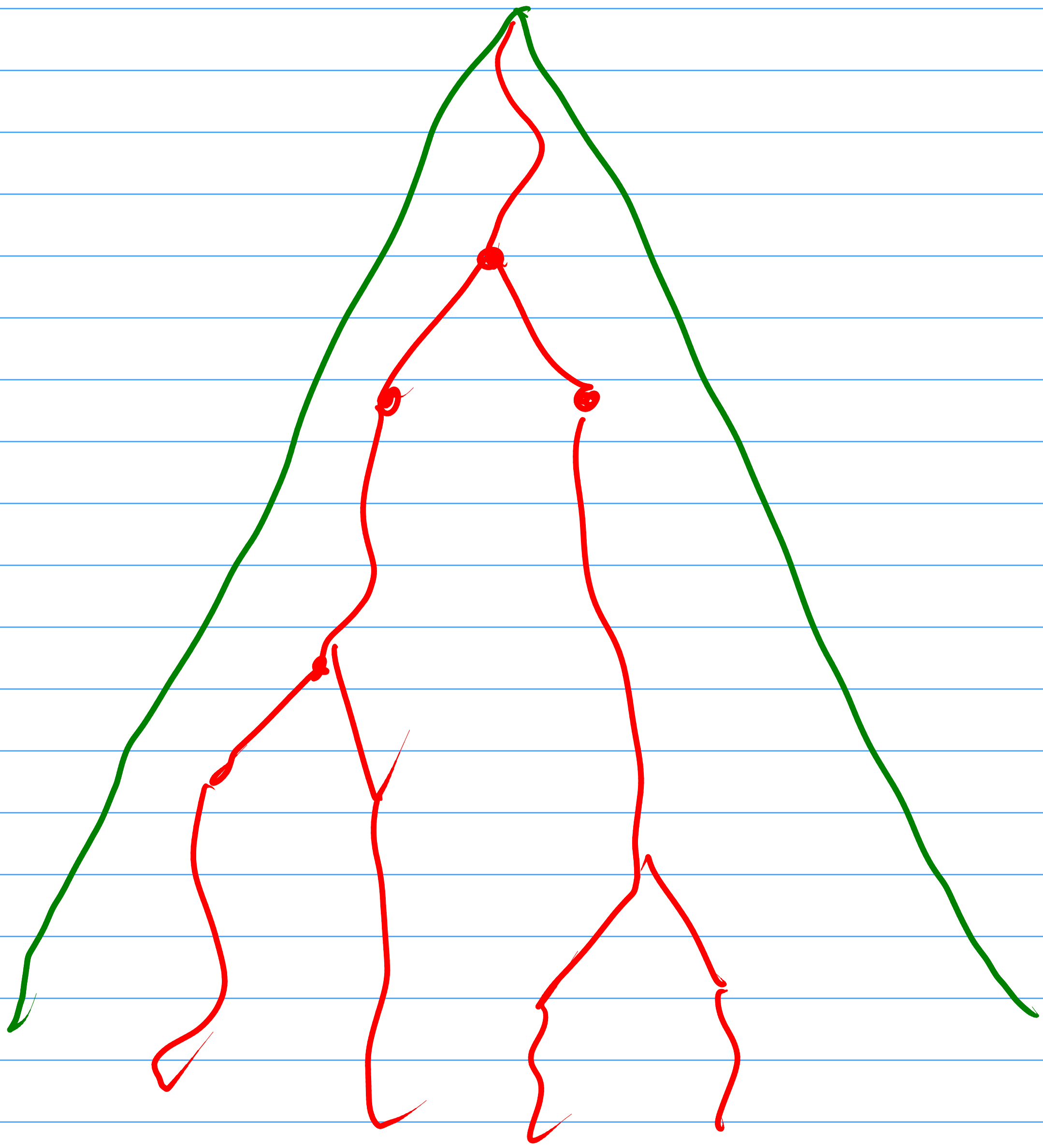
Let $X_{\langle 0 \rangle} = X \cap S_{\alpha_\phi}$

$X_{\langle 1 \rangle} = X \setminus S_{\alpha_\phi}$

How do the paths of

$X_{\langle 0 \rangle}$, $X_{\langle 1 \rangle}$ look like?





So far, we have defined $\{\sigma_\alpha \mid \alpha < \delta\}$, we have defined less than κ many nodes

So there is a branch of this Cantor tree such that below it, there are no σ_α for $\alpha < \delta$

In this way, we find $\{X_n \mid n \in \omega\}$, $\{\eta_n \mid n \in \omega\}$ and τ such that:

1) $\{X_n \mid n \in \omega\} \subseteq \mathcal{I}(A_{\xi_\delta})^+$ is decreasing

2) $X_0 = X$

3) $\tau, \eta_0, \eta_1, \ldots \in 2^{<\omega}$

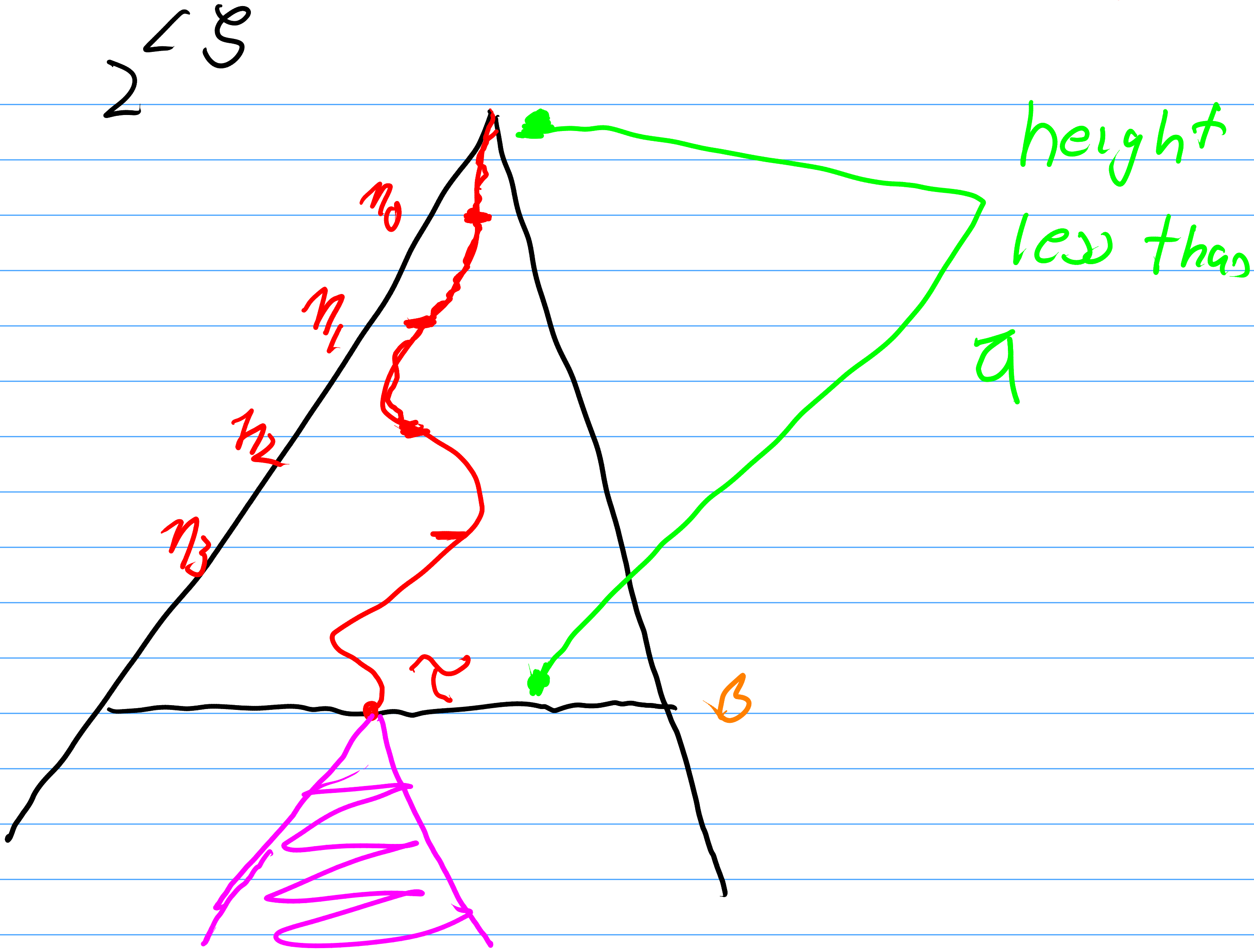
4) $\eta_i \subseteq \eta_{i+1}$

5) $\tau = \bigcup_{i \in \omega} \eta_i$

6) below τ (recall trees grow downwards!) there is no σ_α for $\alpha < \delta$.

7) If $n \in \omega$ and $\xi < \mathrm{dom}(\eta_n)$
$$\Rightarrow X_n \cap S_\xi^{1-\eta_n(\xi)} \in \mathcal{I}(A_{\xi_\delta})$$

$\xi \le \alpha$



Since $\mathcal{I}(A_{<\delta})$ is $\mathfrak{p}^+$, there is $Y \in \mathcal{I}(A_{<\delta})^+$ pseudointersection of $\{X_n \mid n \in \omega\}$ Let $\beta = \mathrm{dom}(\tau)$

Remark: If $\xi < \beta \Rightarrow Y \cap S_{\xi}^{1-\eta(\xi)} \in \mathcal{I}(A)$

For every $\xi < \beta$, let

$F_\xi \in [A_{<\xi}]^{<\omega}$ such that

$Y \cap S_\xi^{1-\eta(\xi)} \subseteq^* \bigcup F_\xi$

Let $W = \{A_\alpha \mid \sigma_\alpha \subseteq \tau\}$

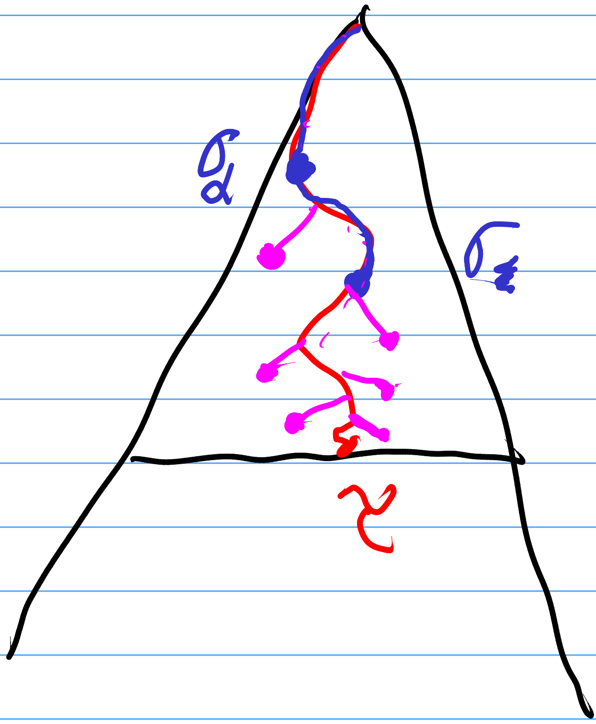
Let $\mathcal{D} = W \cup \bigcup_{\xi < \delta} F_\xi$

Remark $|\mathcal{D}| < \delta \leq \mathfrak{a}$

Let $\mathcal{D} \upharpoonright Y = \{Y \cap A \mid A \in \mathcal{D}\}$

is not MAD so there is

$A \in [Y]^\omega$ with $A \in \mathcal{D}^\perp$



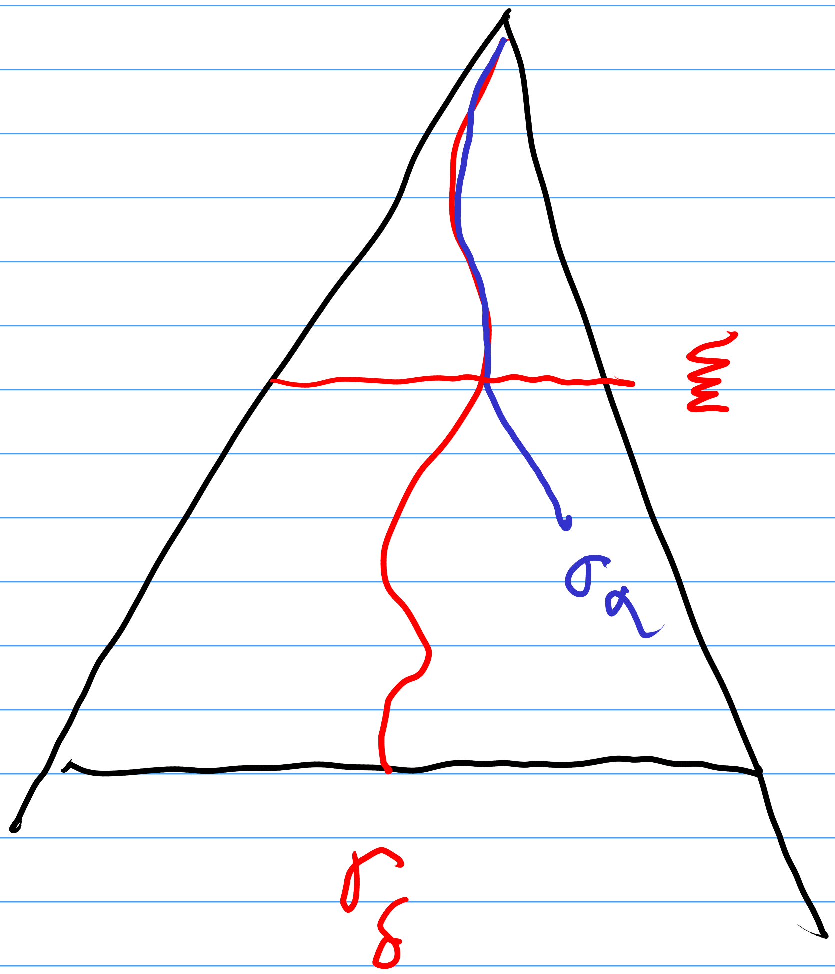
Let $\sigma_\delta = \tau$ $\quad A_\delta = A$

$\boxed{\text{Claim } A_\delta \in \mathcal{A}^{\perp}_{<\delta}}$

Let $\alpha < \delta$

*) If $\alpha \in W$ (if $\sigma_\alpha \subseteq \sigma_\delta$)

$\Rightarrow A_\delta \cap A_\alpha$ is finite



Assume $\sigma_\alpha \nsubseteq \sigma_\delta$.

We also know that $\sigma_\delta \nsubseteq \sigma_\alpha$.

Let $\xi := \Delta(\sigma_\delta, \sigma_\alpha)$

then:

1) $A_\alpha \subseteq^* S_\xi^{\sigma_\alpha(\xi)} = S_\xi^{1-\sigma_\delta(\xi)}$

2) $Y \cap S_\xi^{1-\sigma_\delta(\xi)} \in \mathcal{I}(A)$

in fact $Y \cap S_\xi^{1-\sigma_\delta(\xi)} \subseteq^* \bigcup F_\xi$

$\Rightarrow Y \cap A_\alpha \subseteq^* \bigcup F_\xi$

since $A_\delta \subseteq Y$ is AD with $\bigcup F_\xi$

$\Rightarrow A_\delta \cap A_\alpha$ is finite

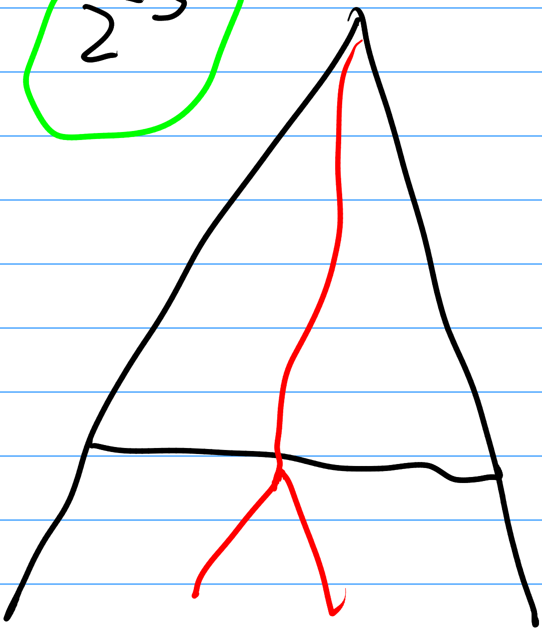
$\square$

Main steps of the proof

We are at step δ

1) $X \in \mathcal{I}(A_{<\delta})^+$

$2^{<\xi}$



Find $\sigma_\delta \in 2^{<\xi}$ such that:

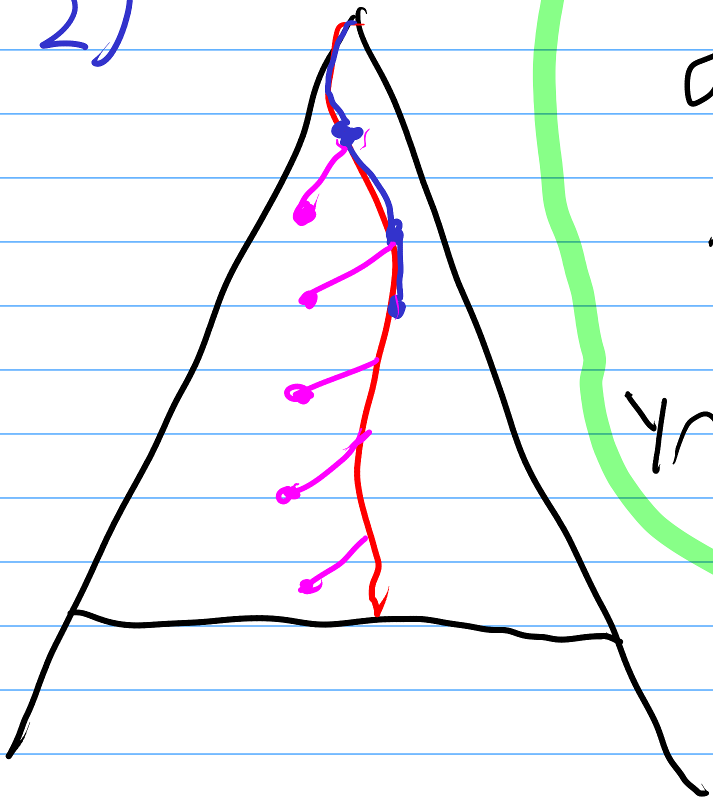
1) There is $Y \subseteq X$, $Y \in \mathcal{I}(A_{<\delta})^+$ "that follows σ_δ"

$$\left(Y \cap S_\xi^{1-\sigma_\delta(\xi)} \in \mathcal{I}(A_{<\delta})\right)$$

This is possible because

$\delta < \xi$

2)



Collect all α such that $\sigma_\alpha \subseteq \sigma_\delta$

And all $F_\xi \in [A_{<\delta}]^{<\omega}$

$$Y \cap S_\xi^{1-\sigma_\delta(\xi)} \subseteq^* \bigcup F_\xi$$

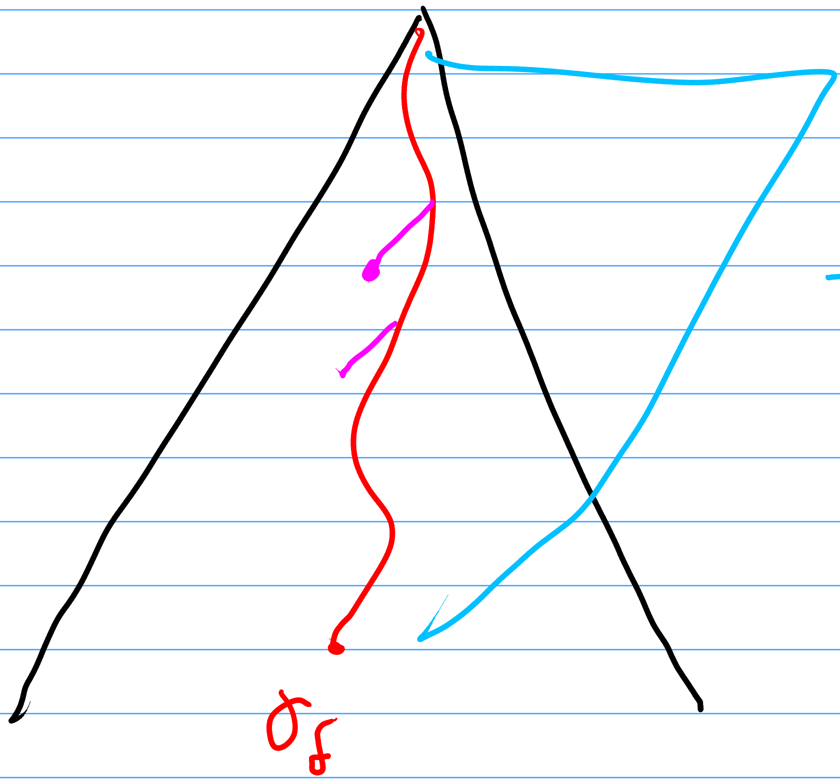
3)

Since $|w| < \delta \leq \mathfrak{a}$

then there is $A_\delta \subseteq Y$ AD with what we collected at step 2

This A_δ will work!

What if $a < g$



there are more than a initial segments

Somehow we need to avoid this situation!

In this context a PCF hypothesis is an assumption on $([\kappa]^\omega, \subseteq^*)$ $\mathrm{cof}(\kappa) > \omega$

Assume $\kappa < \delta$.

$P(\kappa, \delta)$ There is a sequence $\langle U_\alpha \mid \omega \leq \alpha < \delta \rangle$ such that:

a) $U_\alpha \subseteq \alpha$, $|U_\alpha| = \omega$

b) If $X \subseteq \delta$ unbounded of order type κ, then there is $\omega \leq \alpha < \sup(X)$ such that $|U_\alpha \cap X| = \omega$.

It is is possible to show that

If $\kappa < \varsigma$ and $\boxed{\mathrm{cof}([\varsigma]^\omega) = \varsigma}$

$\Rightarrow P(\kappa, \varsigma)$

By induction, it is possible

to prove $\mathrm{cof}([\omega_n]^\omega) = \omega_n$

for $n \in \omega$, $n > 0$

Theorem (Shelah)

If $\kappa < \varsigma$ and $P(\kappa, \varsigma)$

$\Rightarrow$ there is a completely separable MAD family

It is similar to the previous one (much more technical!)

Using the super guessing powers of $\langle U_\alpha \mid \omega \leq \alpha < \varsigma \rangle$ we avoid the

bad case

(where $|\mathcal{D}| \geq \mathfrak{a}$)

$\boxed{\text{corol If } \mathfrak{a} < \aleph_\omega \Rightarrow \text{there is a cumptely sep. MAD family}}$

There are more applications of this method

Weakly tight MAD families

$\mathcal{A}$ is **weakly tight** if

$\forall \{X_n \mid n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+ \Rightarrow \exists$

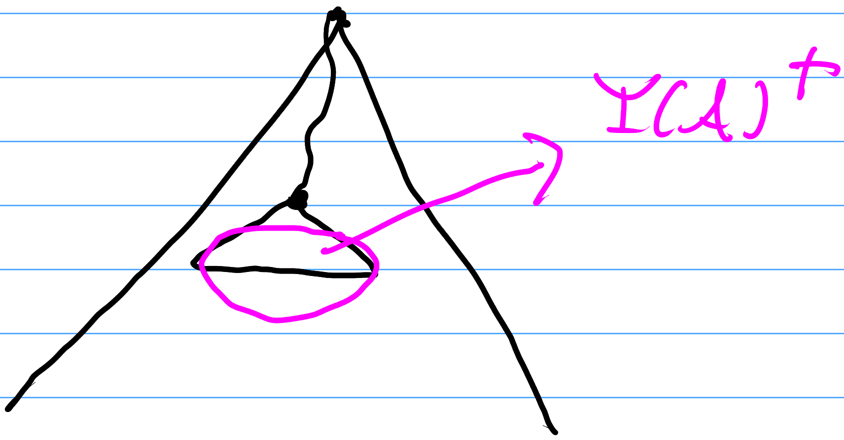
$B \in \mathcal{I}(\mathcal{A}) \left(\exists^\infty n \in \omega \left(|X_n \cap B| = \omega \right) \right)$

Raghavan, Steprāns

$\mathfrak{s} \leq \mathfrak{b} \Rightarrow$ There is a weakly tight MAD family

t-Ramsey MAD families

$\mathcal{A}$ is t-Ramsey if every $\mathcal{I}(\mathcal{A})^+$-branching tree has a branch in $\mathcal{I}(\mathcal{A})^+$.



6. There is a t-Ramsey MAD family

$\mathfrak{a} < \mathfrak{d}$ ✓ $\quad$ $\mathfrak{d} \leq \mathfrak{a} \Rightarrow \mathfrak{s} \leq \mathfrak{a}$

Raghavan has much more results!

If $\mathfrak{c} < \aleph_\omega \Rightarrow$ There is a weakly tight MAD family

Petr Simon, Miroslav Olšák

They use the method to construct compact Frechet spaces whose product is not Frechet