Blowing up the power of a singular cardinal of uncountable cofinality with collapses

Sittinon (New) Jirattikansakul

RIMS Set Theory Workshop

November 18th, 2020

Outline

- Definitions
- Main theorem
- Extenders
- Big Pictures
- Forcings
- Forcings extensions
- Some conclusions

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From the definition above, we have that E is Mitchell below F in the sense that $E \in \mathrm{Ult}(V, F)$.

Main theorem

Theorem (J.)

Given an increasing sequence of cardinals $\langle \kappa_\alpha : \alpha < \eta \rangle$ *where* $\eta < \kappa_0$ *is limit. Let* $\lambda = (\sup_{\alpha<\eta} \kappa_\alpha)^{++}$.

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1. *If* $j_\alpha : V \to M_\alpha = \mathrm{Ult}(V, E_\alpha)$, *we have* $\mathrm{crit}(j_\alpha) = \kappa_\alpha$, $j_\alpha(\kappa_\alpha) \geq \lambda$, ${}^{\kappa_\alpha}M_\alpha \subseteq M_\alpha$ *and* M_α *computes cardinals correctly up to and including* λ.

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Then there is a λ-c.c. forcing extension such that in the generic extension, for limit $\beta < \eta$, $2^{\aleph_\beta} > \aleph_{\beta+1}$ and $2^{\aleph_\eta} = \aleph_{\eta+2}$.

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$A \in E_\alpha(d_\alpha)$ *iff* $\mathrm{mc}_\alpha \in j_\alpha(A)$.

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Lemma

$\mathrm{OB}_\alpha(d_\alpha) \in E_\alpha(d_\alpha)$.

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Recall $\mathrm{mc}_\alpha = \{(j_\alpha(\gamma), \gamma) : \gamma \in d_\alpha\}$. Also $(j_\alpha(\kappa_\alpha), \kappa_\alpha) \in \mathrm{mc}_\alpha$ because $\kappa_\alpha \in d_\alpha$.

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If $d_\alpha \subseteq d'_\alpha$, we have a natural projection $\pi_{d'_\alpha, d_\alpha} : \mu \mapsto \mu \restriction d_\alpha$.

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If $d_\alpha \subseteq d'_\alpha$, we have a natural projection $\pi_{d'_\alpha, d_\alpha} : \mu \mapsto \mu \upharpoonright d_\alpha$. This induces a projection from $E_\alpha(d'_\alpha)$ to $E_\alpha(d_\alpha)$.

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4. $\langle d_\alpha : \alpha < \eta \rangle$ is $\subseteq$-increasing.
5. $\ldots$

Forcing extensions

0 1 2 3

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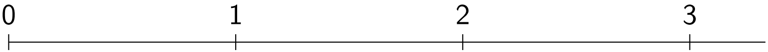

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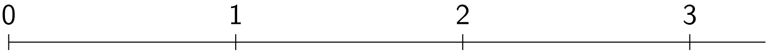

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Direct extension: $q \leq^* p$ if for all α we have

1. $g_\alpha \leq f_\alpha$.
2. B_α projects down to a subset of A_α, meaning $\{\mu \restriction \mathrm{dom}(f_\alpha) : \mu \in B_\alpha\} \subseteq A_\alpha$.

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$q_0 = \langle g_0, B_0, \vec{K}_0 \rangle$ $q_1 = \langle g_1, B_1, \vec{K}_1 \rangle$ $q_2 = \langle g_2, B_2, \vec{K}_2 \rangle$ $q_3 = \langle g_3, B_3, \vec{K}_3 \rangle$

Direct extension: $q \leq^* p$ if for all α we have

1. $g_\alpha \leq f_\alpha$.
2. B_α projects down to a subset of A_α, meaning $\{\mu \restriction \mathrm{dom}(f_\alpha) : \mu \in B_\alpha\} \subseteq A_\alpha$.
3. For $l = 0, 1, 2$, $K^l_\alpha(\mu) \leq H^l_\alpha(\mu \restriction \mathrm{dom}(f_\alpha))$.

Forcing extensions

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$q_0 = \langle t_0, C_0, \vec{L}_0 \rangle$ $q_1 = \langle t_1, C_1, \vec{L}_1 \rangle$ $q_2 = \langle g_2, \lambda_2, \vec{h}_2 \rangle$ $q_3 = \langle f_3, A_3, \vec{H}_3 \rangle$

One-step extension (example): p is pure and $\mu \in A_2$.

Forcing extensions

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One-step extension (example): p is pure and $\mu \in A_2$. One-step extension of p by μ is a condition q such that:

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Forcing extensions

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4. $\vec{h}_2 = \vec{H}_2(\mu)$.
5. $t_0 = f_0 \circ \mu^{-1}$, $t_1 = f_1 \circ \mu^{-1}$, $C_0 = A_0 \circ \mu^{-1}$, $C_1 = A_1 \circ \mu^{-1}$.

Forcing extensions

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$p_0 = \langle f_0, A_0, \vec{H}_0 \rangle$	$p_1 = \langle f_1, A_1, \vec{H}_1 \rangle$	$p_2 = \langle f_2, A_2, \vec{H}_2 \rangle$	$p_3 = \langle f_3, A_3, \vec{H}_3 \rangle$
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- $\kappa_1 < \lambda_2 < \kappa_2$

Forcing extensions

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- $\kappa_1 < \lambda_2 < \kappa_2$.
- $\langle q_0, q_1 \rangle$ will now live in $\mathbb{P}_{\langle E_0 \restriction \lambda_2, E_1 \restriction \lambda_2 \rangle}$.

Forcing extensions

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$p_0 = \langle f_0, A_0, \vec{H}_0 \rangle$	$p_1 = \langle f_1, A_1, \vec{H}_1 \rangle$	$p_2 = \langle f_2, A_2, \vec{H}_2 \rangle$	$p_3 = \langle f_3, A_3, \vec{H}_3 \rangle$
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- $\kappa_1 < \lambda_2 < \kappa_2$.
- $\langle q_0, q_1 \rangle$ will now live in $\mathbb{P}_{\langle E_0 \restriction \lambda_2, E_1 \restriction \lambda_2 \rangle}$.
- $\vec{h}_2 \in \text{Col}(\kappa_1, < g_2(\kappa_2)) \times \text{Col}(g_2(\kappa_2), s_2(g_2(\kappa_2))^+) \times \text{Col}((s_2(g_2(\kappa_2)))^{+3}, < \kappa_2)$.

Forcing extensions

$$\begin{array}{cccc} 0 & 1 & 2 & 3 \\ p_0 = \langle f_0, A_0, \vec{H}_0\rangle & p_1 = \langle f_1, A_1, \vec{H}_1\rangle & p_2 = \langle f_2, A_2, \vec{H}_2\rangle & p_3 = \langle f_3, A_3, \vec{H}_3\rangle \\ q_0 = \langle t_0, C_0, \vec{L}_0\rangle & q_1 = \langle t_1, C_1, \vec{L}_1\rangle & q_2 = \langle g_2, \lambda_2, \vec{h}_2\rangle & q_3 = \langle f_3, A_3, \vec{H}_3\rangle \end{array}$$

- $\kappa_1 < \lambda_2 < \kappa_2$.
- $\langle q_0, q_1\rangle$ will now live in $\mathbb{P}_{\langle E_0 \restriction \lambda_2, E_1 \restriction \lambda_2\rangle}$.
- $\vec{h}_2 \in \mathrm{Col}(\kappa_1, < g_2(\kappa_2)) \times \mathrm{Col}(g_2(\kappa_2), s_2(g_2(\kappa_2))^+) \times \mathrm{Col}((s_2(g_2(\kappa_2)))^{+3}, < \kappa_2)$.
- In particular, a few cardinals in the interval $(\kappa_1, \kappa_2]$ are preserved.

Some conclusions

Let $\overline{\kappa}_\eta = \sup_{\alpha<\eta} \kappa_\alpha$. Then $\lambda = \overline{\kappa}_\eta^{++}$.

- The forcing has the Prikry property.
- Only few cardinals in $(\kappa_\alpha, \kappa_{\alpha+1}]$ are preserved, and hence $\overline{\kappa}_\eta$ is a cardinal, and is equal to $\aleph_\eta$.
- Need a special argument to preserve $\overline{\kappa}_\eta^+$.
- The forcing is λ-c.c., so preserves λ and $\lambda = \aleph_{\eta+2}$ in the extension.
- One can derive a scale on $\overline{\kappa}_\eta$ of length λ. Hence in the extension, $\aleph_{\eta+2} = \lambda = 2^{\overline{\kappa}_\eta} = 2^{\aleph_\eta}$.

Thank you!