Cardinal characteristics associated with families of functions and permutations

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RIMS Set Theory Workshop

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Notations:

- ${}^{B}A$ is the set of all functions from B to A.
- $\mathrm{Sym}(A)$ is the set of all permutations on A.
- $[A]^{\omega} = \{X \subseteq A : |X| = \aleph_0\}$.

Continuum Hypothesis (CH)

There is no cardinal κ such that

$$\aleph_0 < \kappa < \mathfrak{c}.$$

In other words, $\aleph_1 = \mathfrak{c}$.

It is well-known that CH is relatively independent from ZFC.

Some concepts in infinitary combinatorics lead to cardinal characteristics which lie inclusively between $\aleph_1$ and $\mathfrak{c}$. These cardinals are mostly defined on families of infinite sets of natural numbers. We study cardinal characteristics associated with families of functions and permutations on the set of natural numbers.

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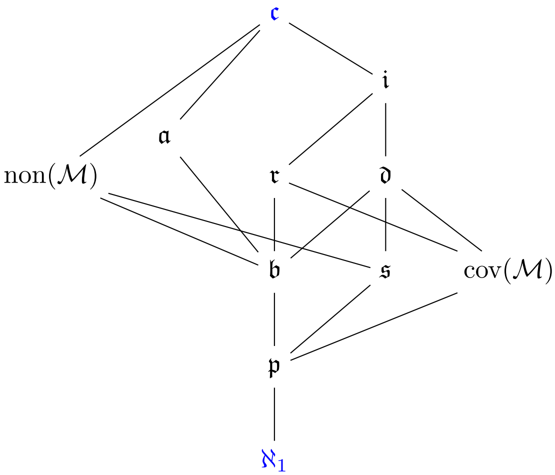


$\mathfrak{c}$
$\mathfrak{i}$
$\mathfrak{a}$
non($\mathcal{M}$)
$\mathfrak{r}$
$\mathfrak{d}$
$\mathfrak{b}$
$\mathfrak{s}$
cov($\mathcal{M}$)
$\mathfrak{p}$
$\aleph_1$

The Cardinals $\mathrm{cov}(\mathcal{M})$ and $\mathrm{non}(\mathcal{M})$

Let $\mathcal{M}$ be the set of meagre subsets of $\mathbb{R}$.

$$\mathrm{cov}(\mathcal{M}) = \min\Big\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{M} \text{ and } \bigcup \mathcal{A} = \mathbb{R}\Big\},$$
$$\mathrm{non}(\mathcal{M}) = \min\Big\{|A| : A \subseteq \mathbb{R} \text{ and } A \notin \mathcal{M}\Big\}.$$

The Cardinal $\mathfrak{a}$

Two sets $X, Y \in [\omega]^\omega$ are *almost disjoint* if $X \cap Y$ is finite.

An infinite family $\mathcal{A} \subseteq [\omega]^\omega$ is an *almost disjoint (a.d.) family* if its members are pairwise almost disjoint. Such a family $\mathcal{A}$ is a *maximal almost disjoint (m.a.d.) family* if it is maximal with respect to the inclusion.

$$\mathfrak{a} = \min\{|\mathcal{A}| : \mathcal{A} \text{ is a m.a.d. family}\}.$$

The Cardinals $\mathfrak{a}_e$ and $\mathfrak{a}_p$

In [12], Zhang extends the concept of almost disjoint family to families of functions and permutations on ω.

Two functions $f, g \in {}^{\omega}\omega$ are *almost disjoint* if $f \cap g$ is finite.

$$\mathfrak{a}_e = \min\{|\mathcal{A}| : \mathcal{A} \subseteq {}^{\omega}\omega \text{ is a m.a.d. family of functions}\},$$
$$\mathfrak{a}_p = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathrm{Sym}(\omega) \text{ is a m.a.d. family of permutations}\}.$$

Known Results

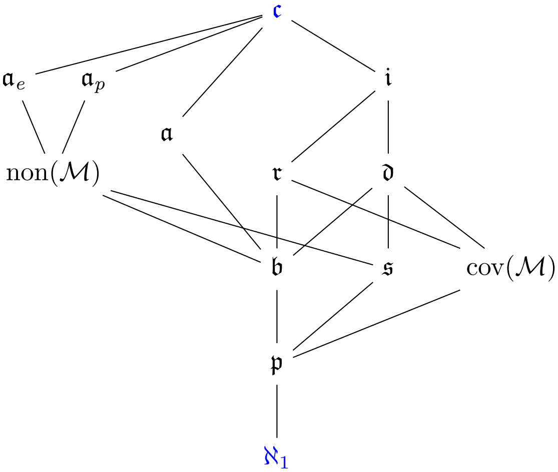
Theorem 1 (Zhang, 1999)

Each of the following statements is relatively consistent with ZFC:

- $\mathfrak{a} < \mathfrak{a}_e$,
- $\mathfrak{a} < \mathfrak{a}_p$.

Theorem 2 (Brendle, Spinas, Zhang, 2000)

- $\mathrm{non}(\mathcal{M}) \leq \mathfrak{a}_e$,
- $\mathrm{non}(\mathcal{M}) \leq \mathfrak{a}_p$.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$\mathfrak{p}$
$\aleph_1$

The Cardinals $\mathfrak{d}$ and $\mathfrak{b}$

For any two functions $f, g \in {}^{\omega}\omega$, we say that g *dominates* f if

$$f(n) \leq g(n) \text{ for all but finitely many } n \in \omega.$$

A family $\mathcal{D} \subseteq {}^{\omega}\omega$ is a *dominating family* if each function in ${}^{\omega}\omega$ is dominated by some member of $\mathcal{D}$.

$$\mathfrak{d} = \min\{|\mathcal{D}| : \mathcal{D} \subseteq {}^{\omega}\omega \text{ is a dominating family}\}.$$

A family $\mathcal{B} \subseteq {}^{\omega}\omega$ is an *unbounded family* if there is no function in ${}^{\omega}\omega$ which dominates every member of $\mathcal{B}$.

$$\mathfrak{b} = \min\{|\mathcal{B}| : \mathcal{B} \subseteq {}^{\omega}\omega \text{ is an unbounded family}\}.$$

New Cardinals $\mathfrak{d}_p$ and $\mathfrak{b}_p$

$$\mathfrak{d}_p = \min\{|\mathcal{D}| : \mathcal{D} \subseteq \mathrm{Sym}(\omega) \text{ is a dominating family}\},$$
$$\mathfrak{b}_p = \min\{|\mathcal{B}| : \mathcal{B} \subseteq \mathrm{Sym}(\omega) \text{ is an unbounded family}\}.$$

Theorem 3

- $\mathfrak{d}_p = \mathfrak{c}$,
- $\mathfrak{b}_p = 2$.

The Cardinals $\mathfrak{s}$ and $\mathfrak{r}$

For any two sets $A, B \in [\omega]^\omega$, we say that A *splits* B if

$$B \cap A \text{ and } B \setminus A \text{ are infinite.}$$

A family $\mathcal{S} \subseteq [\omega]^\omega$ is a *splitting family* if each member of $[\omega]^\omega$ is split by some member of $\mathcal{S}$.

$$\mathfrak{s} = \min\{|\mathcal{S}| : \mathcal{S} \subseteq [\omega]^\omega \text{ is a splitting family}\}.$$

A family $\mathcal{R} \subseteq [\omega]^\omega$ is a *reaping family* if there is no set in $[\omega]^\omega$ which splits every member of $\mathcal{R}$.

$$\mathfrak{r} = \min\{|\mathcal{R}| : \mathcal{R} \subseteq [\omega]^\omega \text{ is a reaping family}\}.$$

New Cardinals $\mathfrak{s}_f$, $\mathfrak{r}_f$, $\mathfrak{s}_p$, and $\mathfrak{r}_p$

$$\mathfrak{s}_f = \min\{|\mathcal{S}| : \mathcal{S} \subseteq {}^{\omega}\omega \text{ is a splitting family}\},$$
$$\mathfrak{r}_f = \min\{|\mathcal{R}| : \mathcal{R} \subseteq {}^{\omega}\omega \text{ is a reaping family}\},$$
$$\mathfrak{s}_p = \min\{|\mathcal{S}| : \mathcal{S} \subseteq \mathrm{Sym}(\omega) \text{ is a splitting family}\},$$
$$\mathfrak{r}_p = \min\{|\mathcal{R}| : \mathcal{R} \subseteq \mathrm{Sym}(\omega) \text{ is a reaping family}\}.$$

Fact. $\mathfrak{s} \leq \mathrm{non}(\mathcal{M})$ and $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{r}$.

Theorem 4 (S.)

- $\mathfrak{s}_f = \mathrm{non}(\mathcal{M}) = \mathfrak{s}_p$,
- $\mathfrak{r}_f = \mathrm{cov}(\mathcal{M}) \leq \mathfrak{r}_p$.

$\mathfrak{s}_f = \mathrm{non}(\mathcal{M})$ and $\mathfrak{r}_f = \mathrm{cov}(\mathcal{M})$

Proof. Use the fact that

$$\mathrm{cov}(\mathcal{M}) = \min\{|\mathcal{C}| : \mathcal{C} \subseteq {}^\omega\omega \wedge \neg\exists f \in {}^\omega\omega\ \forall g \in \mathcal{C}\ [f \cap g \text{ is infinite}]\}, \text{ and}$$
$$\mathrm{non}(\mathcal{M}) = \min\{|\mathcal{C}| : \mathcal{C} \subseteq {}^\omega\omega \wedge \forall f \in {}^\omega\omega\ \exists g \in \mathcal{C}\ [f \cap g \text{ is infinite}]\}.$$

$\mathrm{cov}(\mathcal{M}) \leq \mathfrak{r}_p$

Proof. Use the fact that $\mathrm{cov}(\mathcal{M}) = \mathfrak{m}_{ctbl}$ and consider the poset $\mathrm{Fn}_{\text{1-1}}(\omega, \omega)$.

$\mathrm{non}(\mathcal{M}) = \mathfrak{s}_p$

Proof. Consider $\mathrm{Sym}(\omega)$ as a Polish space.

Try to connect a set of functions and a set of permutations.

The Cardinal $\mathfrak{i}$

An infinite family $\mathcal{I} \subseteq [\omega]^\omega$ is an *independent family* if, for any two finite disjoint sets $\mathcal{A}, \mathcal{B} \subseteq \mathcal{I}$, $\bigcap \mathcal{A} \setminus \bigcup \mathcal{B}$ is infinite (here $\bigcap \emptyset = \omega$).

Such a family $\mathcal{I}$ is a *maximal independent family* if it is maximal with respect to the inclusion.

$$\mathfrak{i} = \min\{|\mathcal{I}| : \mathcal{I} \subseteq [\omega]^\omega \text{ is a maximal independent family}\}.$$

New Cardinals $\mathfrak{i}_f$ and $\mathfrak{i}_p$

$$\mathfrak{i}_f = \min\{|\mathcal{I}| : \mathcal{I} \subseteq {}^{\omega}\omega \text{ is a maximal independent family}\},$$

$$\mathfrak{i}_p = \min\{|\mathcal{I}| : \mathcal{I} \subseteq \mathrm{Sym}(\omega) \text{ is a maximal independent family}\}.$$

Theorem 5 (S. and Vejjajiva, 2020)

- $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{d} \leq \mathfrak{i}_f \leq \mathfrak{i}$,
- $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{i}_p \leq \mathfrak{i}$.

Note that, for $\mathfrak{a}_e$ and $\mathfrak{a}_p$,

- ZFC $\nvdash \mathfrak{a}_e \leq \mathfrak{a}$,
- ZFC $\nvdash \mathfrak{a}_p \leq \mathfrak{a}$.

$\boxed{\mathrm{cov}(\mathcal{M}) \leq \mathfrak{i}_f, \mathfrak{i}_p}$

Proof. Use the fact that $\mathrm{cov}(\mathcal{M}) = \mathfrak{m}_{ctbl}$ and consider the posets $\mathrm{Fn}(\omega, \omega)$ and $\mathrm{Fn}_{1-1}(\omega, \omega)$.

$\boxed{\mathfrak{i}_f, \mathfrak{i}_p \leq \mathfrak{i}}$

Proof. Given an independent family $\mathcal{I} \subseteq [\omega]^\omega$ with $\aleph_0 \leq |\mathcal{I}| < \mathfrak{i}_f$. Convert $\mathcal{I}$ to an independent family of functions $\mathcal{C} \subseteq {}^\omega\omega$ with the same size as $\mathcal{I}$. We get a witness h for the nonmaximality of $\mathcal{C}$, and then convert h to a set H which is a witness for the nonmaximality of $\mathcal{I}$. Similarly for $\mathfrak{i}_p$.

$\boxed{\mathfrak{d} \leq \mathfrak{i}_f}$

Proof. Given an independent family $\mathcal{I} \subseteq {}^{\omega}\omega$ such that $\aleph_1 \leq |\mathcal{I}| < \mathfrak{d}$.
Take a model M of (a large fragment of) ZFC with $|M| = |\mathcal{I}|$ and $\mathcal{I} \in M$.

Construct a sequence $\{n_k : k < \omega\} \subseteq \omega$ with $n_0 = 0$ so that for any $g \in M \cap {}^{\omega}\omega$ there are infinitely many k such that $g(n_k) < n_{k+1}$.
(This part uses the assumption $|M| < \mathfrak{d}$.)

Let $\{f_k : k < \omega\} \subseteq \mathcal{I}$ be a sequence in M without repetitions. Define

$$h = \bigcup_{k<\omega} f_k \restriction [n_k, n_{k+1}).$$

Then h is a witness for the nonmaximality of $\mathcal{I}$.
(This part uses the assumption that the sequence of f_k's lies in M.)

$\boxed{\mathfrak{d} \leq \mathfrak{i}_f}$

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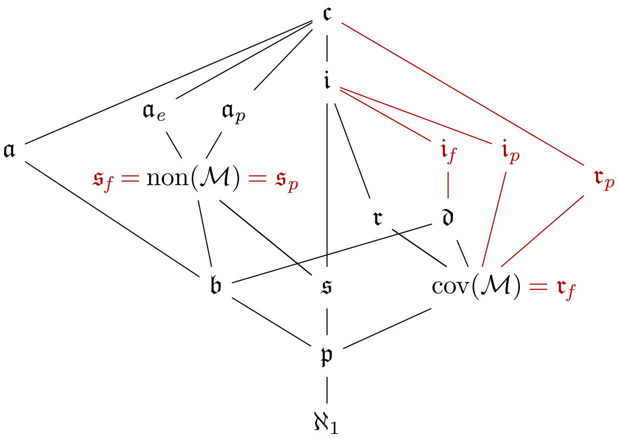
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$\mathfrak{s}$
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$\mathfrak{p}$
$\aleph_1$

Open Problems

1. Is $\mathfrak{r}_p = \mathrm{cov}(\mathcal{M})$?
2. Does $\mathfrak{d} \leq \mathfrak{i}_p$ hold?
3. Is each of the following statements consistent with ZFC?
    - $\mathrm{cov}(\mathcal{M}) < \mathfrak{i}_p$
    - $\mathfrak{i}_f < \mathfrak{i}$
    - $\mathfrak{i}_p < \mathfrak{i}$
4. Are any strict inequalities between $\mathfrak{i}_f$ and $\mathfrak{i}_p$ consistent with ZFC? (Any possible strict inequalities between $\mathfrak{a}_e$ and $\mathfrak{a}_p$ is still unknown.)

References

1. Bartoszyński, T.: Combinatorial aspects of measure and category, *Fundamenta Mathematicae* **127**(3), 225–239 (1987).
2. Bell, M.G..: On the combinatorial principle P(c), *Fundamenta Mathematicae* **114**, 149–157 (1981).
3. Blass, A.: Combinatorial Cardinal Characteristics of the Continuum, *Handbook of set theory* (M. Foreman and A. Kanamori, editors), vol. 1, Springer, 395–490 (2010).
4. Blass, A., Hyttinen, T., Zhang, Y.: *Mad families and their neighbors*, preprint, (2007).
5. Brendle, J., Spinas, O., Zhang, Y.: Uniformity of the Meager Ideal and Maximal Cofinitary Groups, *Journal of Algebra* **232**, 209–225 (2000).
6. Enderton, H.B.: *A Mathematical Introduction to Logic*, Academic Press (2001).

7. Fremlin, D.H., Shelah, S.: On partitions of the real line, *Israel Journal of Mathematics* **32**, 299–304 (1979).
8. Halbeisen, L.J.: *Combinatorial Set Theory: With a Gentle Introduction to Forcing*, Springer, 2017.
9. Kunen, K.: *Set Theory*, Studies in Logic: Mathematical Logic and Foundations, Vol. 34, Colledge Publication, (2011).
10. Shelah, S.: Remarks on some cardinal invariants of the continuum, in *Open Problems in Topology* (J. van Mill and G. Reed, eds.), North-Holland, Amsterdam, 217–218 (1990).
11. Sonpanow, N., Vejjajiva, P.: Independent families of functions and permutations, *Mathematical Logic Quarterly* **66**, 311–315 (2020).
12. Zhang, Y.: On a Class of M.A.D. Families, *The Journal of Symbolic Logic* **64**(2), 737–746 (1999).
13. Zhang, Y.: Adjoining almost disjoint permutations, *Mathematical Logic Quarterly* **48**, 189–193 (2002).

THANK YOU