Cardinal characteristics associated with families of functions and permutations

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RIMS Set Theory Workshop

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Notations:

- $^B A$ is the set of all functions from $B$ to $A$.
- $\text{Sym}(A)$ is the set of all permutations on $A$.
- $[A]^\omega = \{ X \subseteq A : |X| = \aleph_0 \}$.
Continuum Hypothesis (CH)

There is no cardinal $\kappa$ such that

$$\aleph_0 < \kappa < c.$$ 

In other words, $\aleph_1 = c$.

It is well-known that CH is relatively independent from ZFC.

Some concepts in infinitary combinatorics lead to cardinal characteristics which lie inclusively between $\aleph_1$ and $c$. These cardinals are mostly defined on families of infinite sets of natural numbers. We study cardinal characteristics associated with families of functions and permutations on the set of natural numbers.
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Cardinals associated with functions and permutations

\[ \aleph_1 \]

non(\(\mathcal{M}\))

\[ \text{cov}(\mathcal{M}) \]

\(\aleph_1\)
The Cardinals $\text{cov}(\mathcal{M})$ and $\text{non}(\mathcal{M})$

Let $\mathcal{M}$ be the set of meagre subsets of $\mathbb{R}$.

\[
\text{cov}(\mathcal{M}) = \min \left\{ |A| : A \subseteq \mathcal{M} \text{ and } \bigcup A = \mathbb{R} \right\},
\]

\[
\text{non}(\mathcal{M}) = \min \left\{ |A| : A \subseteq \mathbb{R} \text{ and } A \notin \mathcal{M} \right\}.
\]
Two sets $X, Y \in [\omega]^\omega$ are \textit{almost disjoint} if $X \cap Y$ is finite.

An infinite family $A \subseteq [\omega]^\omega$ is an \textit{almost disjoint (a.d.) family} if its members are pairwise almost disjoint. Such a family $A$ is a \textit{maximal almost disjoint (m.a.d.) family} if it is maximal with respect to the inclusion.

$$\alpha = \min\{|A| : A \text{ is a m.a.d. family}\}.$$
The Cardinals $a_e$ and $a_p$

In [12], Zhang extends the concept of almost disjoint family to families of functions and permutations on $\omega$.

Two functions $f, g \in \omega\omega$ are almost disjoint if $f \cap g$ is finite.

\[
a_e = \min\{|A| : A \subseteq \omega\omega \text{ is a m.a.d. family of functions}\},
\]
\[
a_p = \min\{|A| : A \subseteq \text{Sym}(\omega) \text{ is a m.a.d. family of permutations}\}.
\]
Known Results

Theorem 1 (Zhang, 1999)

Each of the following statements is relatively consistent with ZFC:

- $\alpha < \alpha_e$,
- $\alpha < \alpha_p$.

Theorem 2 (Brendle, Spinas, Zhang, 2000)

- $\text{non}(\mathcal{M}) \leq \alpha_e$,
- $\text{non}(\mathcal{M}) \leq \alpha_p$. 
Cardinals associated with functions and permutations

\[ a_e, a_p, \text{non}(M), a, i, \text{cov}(M), p, N_1 \]
The Cardinals $\mathfrak{d}$ and $\mathfrak{b}$

For any two functions $f, g \in \omega \omega$, we say that $g$ dominates $f$ if

$$f(n) \leq g(n) \text{ for all but finitely many } n \in \omega.$$ 

A family $D \subseteq \omega \omega$ is a dominating family if each function in $\omega \omega$ is dominated by some member of $D$.

$$\mathfrak{d} = \min\{|D| : D \subseteq \omega \omega \text{ is a dominating family}\}.$$ 

A family $B \subseteq \omega \omega$ is an unbounded family if there is no function in $\omega \omega$ which dominates every member of $B$.

$$\mathfrak{b} = \min\{|B| : B \subseteq \omega \omega \text{ is an unbounded family}\}.$$
New Cardinals $\mathfrak{d}_p$ and $\mathfrak{b}_p$

\[ \mathfrak{d}_p = \min\{|D| : D \subseteq \text{Sym}(\omega) \text{ is a dominating family}\}, \]
\[ \mathfrak{b}_p = \min\{|B| : B \subseteq \text{Sym}(\omega) \text{ is an unbounded family}\}. \]

**Theorem 3**

- $\mathfrak{d}_p = c$,
- $\mathfrak{b}_p = 2$. 
The Cardinals $s$ and $r$

For any two sets $A, B \in [\omega]^{\omega}$, we say that $A$ splits $B$ if

$$B \cap A \text{ and } B \setminus A \text{ are infinite.}$$

A family $S \subseteq [\omega]^{\omega}$ is a splitting family if each member of $[\omega]^{\omega}$ is split by some member of $S$.

$$s = \min\{|S| : S \subseteq [\omega]^{\omega} \text{ is a splitting family}\}.$$  

A family $R \subseteq [\omega]^{\omega}$ is a reaping family if there is no set in $[\omega]^{\omega}$ which splits every member of $R$.

$$r = \min\{|R| : R \subseteq [\omega]^{\omega} \text{ is a reaping family}\}.$$
New Cardinals $s_f$, $r_f$, $s_p$, and $r_p$

\[
s_f = \min\{|S| : S \subseteq \omega \omega \text{ is a splitting family}\},
\]
\[
r_f = \min\{|R| : R \subseteq \omega \omega \text{ is a reaping family}\},
\]
\[
s_p = \min\{|S| : S \subseteq \text{Sym}(\omega) \text{ is a splitting family}\},
\]
\[
r_p = \min\{|R| : R \subseteq \text{Sym}(\omega) \text{ is a reaping family}\}.
\]

**Fact.** $s \leq \text{non}(\mathcal{M})$ and $\text{cov}(\mathcal{M}) \leq r$.

**Theorem 4 (S.)**

- $s_f = \text{non}(\mathcal{M}) = s_p$,
- $r_f = \text{cov}(\mathcal{M}) \leq r_p$. 
\[ s_f = \text{non}(\mathcal{M}) \text{ and } r_f = \text{cov}(\mathcal{M}) \]

**Proof.** Use the fact that
\[
\text{cov}(\mathcal{M}) = \min\{|C| : C \subseteq \omega \land \neg \exists f \in \omega \forall g \in C \ [f \cap g \text{ is infinite}]\}, \text{ and }
\text{non}(\mathcal{M}) = \min\{|C| : C \subseteq \omega \land \forall f \in \omega \exists g \in C \ [f \cap g \text{ is infinite}]\}.
\]

\[ \text{cov}(\mathcal{M}) \leq r_p \]

**Proof.** Use the fact that \( \text{cov}(\mathcal{M}) = m_{ctbl} \) and consider the poset \( \text{Fn}_{1-1}(\omega, \omega) \).

\[ \text{non}(\mathcal{M}) = s_p \]

**Proof.** Consider \( \text{Sym}(\omega) \) as a Polish space.
Try to connect a set of functions and a set of permutations.
The Cardinal \( i \)

An infinite family \( \mathcal{I} \subseteq [\omega]^\omega \) is an \textit{independent family} if, for any two finite disjoint sets \( A, B \subseteq \mathcal{I} \), \( \bigcap A \setminus \bigcup B \) is infinite (here \( \bigcap \emptyset = \omega \)).

Such a family \( \mathcal{I} \) is a \textit{maximal independent family} if it is maximal with respect to the inclusion.

\[
i = \min\{|\mathcal{I}| : \mathcal{I} \subseteq [\omega]^\omega \text{ is a maximal independent family}\}.
\]
New Cardinals $i_f$ and $i_p$

\[ i_f = \min\{|I| : I \subseteq \omega \omega \text{ is a maximal independent family}\}, \]

\[ i_p = \min\{|I| : I \subseteq \text{Sym}(\omega) \text{ is a maximal independent family}\}. \]

Theorem 5 (S. and Vejjajiva, 2020)

- $\text{cov}(\mathcal{M}) \leq d \leq i_f \leq i$,
- $\text{cov}(\mathcal{M}) \leq i_p \leq i$.

Note that, for $a_e$ and $a_p$,

- $\text{ZFC} \not\models a_e \leq a$,
- $\text{ZFC} \not\models a_p \leq a$. 

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Cardinals associated with functions and permutations
\[ \text{cov}(\mathcal{M}) \leq i_f, i_p \]

**Proof.** Use the fact that \( \text{cov}(\mathcal{M}) = m_{ctbl} \) and consider the posets \( \text{Fn}(\omega, \omega) \) and \( \text{Fn}_{1-1}(\omega, \omega) \).

\[ i_f, i_p \leq i \]

**Proof.** Given an independent family \( I \subseteq [\omega]^\omega \) with \( \aleph_0 \leq |I| < i_f \). Convert \( I \) to an independent family of functions \( C \subseteq \omega \omega \) with the same size as \( I \). We get a witness \( h \) for the nonmaximality of \( C \), and then convert \( h \) to a set \( H \) which is a witness for the nonmaximality of \( I \). Similarly for \( i_p \).
Proof. Given an independent family $\mathcal{I} \subseteq \omega\omega$ such that $\aleph_1 \leq |\mathcal{I}| < d$.

Take a model $M$ of (a large fragment of) ZFC with $|M| = |\mathcal{I}|$ and $\mathcal{I} \in M$.

Construct a sequence $\{n_k : k < \omega\} \subseteq \omega$ with $n_0 = 0$ so that for any $g \in M \cap \omega\omega$ there are infinitely many $k$ such that $g(n_k) < n_{k+1}$.

(This part uses the assumption $|M| < d$.)

Let $\{f_k : k < \omega\} \subseteq \mathcal{I}$ be a sequence in $M$ without repetitions. Define

$$h = \bigcup_{k<\omega} f_k \upharpoonright [n_k, n_{k+1}).$$

Then $h$ is a witness for the nonmaximality of $\mathcal{I}$.

(This part uses the assumption that the sequence of $f_k$’s lies in $M$.)
\[ d \leq i_f \]

**Proof.** Given an independent family \( \mathcal{I} \subseteq \omega \omega \) such that \( \aleph_1 \leq |\mathcal{I}| < d \).

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\[ s_f = \text{non}(\mathcal{M}) = s_p \]
\[ \text{cov}(\mathcal{M}) = r_f \]
\[ \aleph_1 \]
Open Problems

1. Is $r_p = \text{cov}(\mathcal{M})$?

2. Does $\mathfrak{d} \leq i_p$ hold?

3. Is each of the following statements consistent with ZFC?
   - $\text{cov}(\mathcal{M}) < i_p$
   - $i_f < i$
   - $i_p < i$

4. Are any strict inequalities between $i_f$ and $i_p$ consistent with ZFC? (Any possible strict inequalities between $a_e$ and $a_p$ is still unknown.)
References


THANK YOU