Highly connected Ramsey theory

Chris Lambie-Hanson

Department of Mathematics and Applied Mathematics
Virginia Commonwealth University

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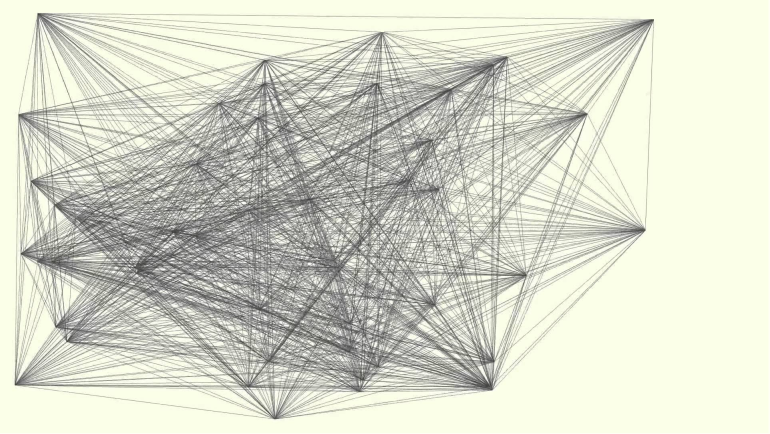
I. Highly connected partition relations



Classical partition relations

Recall the Hungarian notation for partition relations: If λ, μ, and ν are cardinals and k is a natural number, then

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is the assertion that, for every function $c : [\nu]^k \rightarrow \lambda$, there is $H \subseteq \nu$ of size μ such that $c \restriction [H]^k$ is constant.

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- An uncountable cardinal κ is weakly compact if and only if

$$\kappa \to (\kappa)^2_2.$$

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- Define $d : [\mathbb{R}]^2 \to 2$ as follows. Fix a well-ordering $\prec$ on $\mathbb{R}$ and let $d(x, y) = 0$ if $\prec$ agrees with the usual ordering of $\mathbb{R}$ on the order of x and y, and $d(x, y) = 1$ otherwise. Then d witnesses

$$\mathfrak{c} \not\to (\aleph_1)^2_2.$$

Connectedness

The relation $\nu \to (\mu)^2_\lambda$ can be phrased in graph-theoretic language: Whenever the edges of the complete graph on ν-many vertices are colored with λ-many colors, we can find a complete monochromatic subgraph of size μ.

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Definition

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph.

1. $\mathcal{G}$ is connected if, for all $u, v \in \mathcal{V}$, there are $u_0, u_1, \ldots, u_n \in \mathcal{V}$ such that $u_0 = u$, $u_n = v$, and, for all $i < n$, $\{u_i, u_{i+1}\} \in \mathcal{E}$.

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2. $\mathcal{G}$ is κ-connected if it is connected and remains connected after removing any fewer than κ-many vertices.

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Note: A finite graph is highly connected if and only if it is complete, so the relation

$$\nu \to_{hc} (\mu)^2_\lambda$$

can be seen as a genuine generalization of the classical finite Ramsey partition relations.

Warm-up Exercise

Proposition (Bergfalk-Hrušák-Shelah '20)

If ν is an infinite cardinal and k is a natural number, then $\nu \to_{hc} (\nu)^2_k$.

Proof: Fix $c : [\nu]^2 \to k$ and a uniform ultrafilter $\mathcal{U}$ on ν. For $\alpha < \nu$, we can a set $X_\alpha \in \mathcal{U}$ and $i_\alpha < k$ s.t. for all $\beta \in X_\alpha$, $\beta > \alpha$ and $c(\alpha, \beta) = i_\alpha$.
We can now find a set $X \in \mathcal{U}$ and a $i < k$ s.t. for all $\alpha \in X$, $i_\alpha = i$.

<u>Claim</u> $(X, [X]^2 \cap c^{-1}\{i\})$ is highly connected.

<u>Pf</u> For all $\alpha < \beta$ in X, for all $\gamma \in X \cap X_\alpha \cap X_\beta$ then we have

$\square$

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Proof:
A similar proof yields the following:

Proposition

If κ is strongly compact, $\lambda < \kappa$, and $\mathrm{cf}(\nu) \geq \kappa$, then $\nu \rightarrow_{hc} (\nu)^2_\lambda$.

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- for all $\mu \geq 3$, $\lambda^+ \not\rightarrow_{hc} (\mu)^2_\lambda$;

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- for all $\mu \geq 3$, $\lambda^+ \not\rightarrow_{hc} (\mu)^2_\lambda$;
- $2^\lambda \not\rightarrow_{hc} (2^\lambda)^2_\lambda$.

Consistent positive results

Theorem (Bergfalk-Hrušák-Shelah '20)

If the existence of a weakly compact cardinal is consistent, then it is consistent that

$$2^{\omega_1} \to_{hc} (2^{\omega_1})^2_\omega.$$

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Theorem (Hrušák-Shelah '2X)

If the existence of a measurable cardinal is consistent, then it is consistent that

$$\omega_2 \rightarrow_{hc} (\omega_2)^2_\omega.$$

Motivating questions

1. What effect do other well-known compactness/incompactness principles (e.g. PFA, square principles) have on highly connected partition relations?

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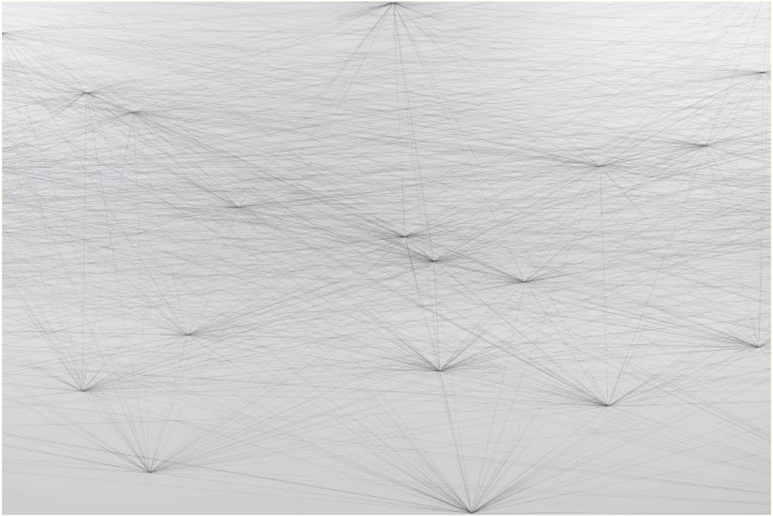
2. Are any nontrivial positive consistency results possible at the level of $\mathfrak{c}$?

Counterexamples from squares



Square bracket relations

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The partition $\nu \to_{hc} [\mu]^2_{\lambda,\kappa}$ (resp. $\nu \to [\mu]^2_{\lambda,<\kappa}$) is the assertion that, for every coloring $c : [\nu]^2 \to \lambda$, there is $H \subseteq \nu$ of size μ and a highly connected graph (H, E) such that $|c``E| \leq \kappa$ (resp. $|c``E| < \kappa$).

Definition (Todorcevic)

Suppose that ν is a regular uncountable cardinal. Then $\square(\nu)$ is the assertion that there is a sequence $\vec{C} = \langle C_\alpha \mid \alpha < \nu \rangle$ such that, for all limit ordinals $\beta < \nu$:

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1. C_β is a club in β;
2. (Coherence) for all $\alpha \in \mathrm{Lim}(C_\beta)$, we have $C_\alpha = C_\beta \cap \alpha$;
3. (Nontriviality) there is no club D in ν such that, for all $\alpha \in \mathrm{Lim}(D)$, we have $C_\alpha = D \cap \alpha$.

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Theorem (Jensen)

If ν is a regular uncountable cardinal and $\square(\nu)$ fails, then ν is weakly compact in L.

Negative results from square

Theorem (LH '20)

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Theorem (LH '20)

If μ is a singular cardinal and $\square_\mu$ holds, then

$$\mu^+ \not\to_{hc} [\mu]^2_{\mathrm{cf}(\mu),<\mathrm{cf}(\mu)}.$$

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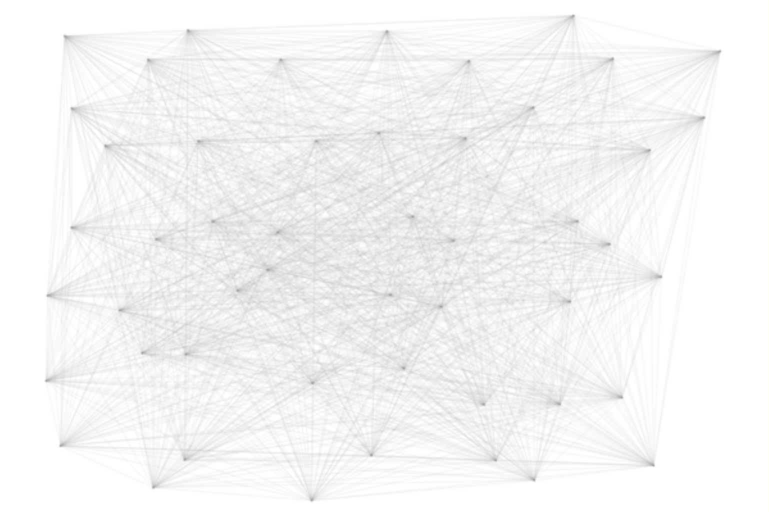
Step 2: Prove that a subadditive unbounded function $c : [\nu]^2 \to \lambda$ witnesses the failure of $\nu \to_{hc} [\nu]^2_{\lambda, <\lambda}$.

A positive result at $\mathfrak{c}$



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Theorem (LH '20)

Suppose that ν is a weakly compact cardinal and $\mathbb{P}$ is the poset to add ν-many Cohen reals. Then, in $V^{\mathbb{P}}$, for all $\lambda < \nu$ we have

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Corollary (LH '20)

The following are equiconsistent over ZFC*:*

1. *There exists a weakly compact cardinal.*
2. $\mathfrak{c} \rightarrow_{hc} [\mathfrak{c}]^2_{\lambda,2}$.

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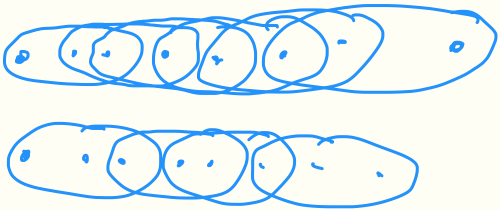

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Theorem (LH '20)

1. *After adding a weakly compact number of Cohen reals, for all $\lambda < \mathfrak{c}$, we have $\mathfrak{c} \to_{hc} [\mathfrak{c}]^k_{\lambda,k}$.*
2. *In* ZFC, *we have $\mathfrak{c} \not\to_{hc} [\mathfrak{c}]^k_{\lambda,k-1}$.*

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Remaining speculations

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Also, to construct consistent counterexamples, we would want to develop the theory of "subadditive unbounded functions" $f : [\nu]^k \to \lambda$.

References

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All artwork by Sol LeWitt

Thank you!