Transversal of full outer measure

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Forcing with σ-ideal

Theorem

Let $\mathbb{P}$ be a forcing. Suppose there are $\langle \mathbb{Q}_n : n \geq 1 \rangle$ and $\langle p_n : n \geq 1 \rangle$ satisfying (1)-(4) below. Then forcing with a σ-ideal cannot be isomorphic to $\mathbb{P}$.

(1) *For every $n \geq 1$, $p_n \in \mathbb{P}$ and $\mathbb{Q}_n \lessdot \mathbb{P}_{\leq p_n}$.*

(2) *$\bigcup\{\mathbb{Q}_n : n \geq 1\}$ is dense in $\mathbb{P}$.*

(3) *Each $\mathbb{Q}_n$ is isomorphic to random forcing.*

(4) *$\mathbb{P}$ adds a Cohen real.*

New reals

From now on, we fix $\mathbb{P}$, $\langle \mathbb{Q}_n : n \geq 1 \rangle$ and $\langle p_n : n \geq 1 \rangle$ satisfying clauses **(1)-(3)** above.

Lemma

Suppose $\mathbb{Q} \lessdot \mathbb{P}$ is atomless. Then forcing with $\mathbb{Q}$ adds a new real.

Proof: Since $\mathbb{Q}$ satisfies ccc, it is enough to show that every $\mathbb{Q}$-generic extension contains a new ω-sequence of members of V. Since random forcing is σ-linked, $\bigcup \mathbb{Q}_n$ is σ-linked. Since $\bigcup \mathbb{Q}_n$ is dense in $\mathbb{P}$, it follows that $\mathbb{Q}$ is also σ-linked. Let $\mathbb{Q} = \bigcup \{L_n : n < \omega\}$ where each L_n has pairwise compatible conditions in $\mathbb{Q}$.

Towards a contradiction, suppose $p \in \mathbb{Q}$ forces that no such sequence appears in the extension. Let α be the least ordinal such that for some $\nu \in V^{\mathbb{Q}}$ and $q \in \mathbb{Q}$, $q \leq p$ and $q \Vdash_{\mathbb{Q}} \nu : \alpha \to V \wedge \nu \notin V$. It is clear that α is regular uncountable. Fix $q \leq p$ and $\nu \in V^{\mathbb{Q}}$ such that $q \Vdash_{\mathbb{Q}} \nu : \alpha \to V \wedge \nu \notin V$.

New reals

For each $\beta < \alpha$, fix a maximal antichain $A_\beta \subseteq \mathbb{Q}$ such that for every $p \in A_\beta$ there exists $\nu_{\beta,p} \in V$ such that $p \Vdash_{\mathbb{Q}} \nu \restriction \beta = \nu_{\beta,p}$. Choose $n < \omega$ such that $W = \{\beta < \alpha : A_\beta \cap L_n \neq \emptyset\}$ has size α. For each $\beta \in W$, fix $p_\beta \in A_\beta \cap L_n$.

Note that $\{\nu_{\beta,p_\beta} : \beta \in W\}$ has pairwise compatible sequences. Let $\nu_\star$ be their union. Since $\mathbb{Q}$ satisfies ccc, there exists $q' \leq q$ such that $q' \Vdash_{\mathbb{Q}} |G_{\mathbb{Q}} \cap \{q_\beta : \beta \in W\}| = \alpha$. But $q' \Vdash_{\mathbb{Q}} \nu = \nu_\star \in V$ which is impossible. ☕

Infinitely often equal or random

Lemma

Suppose τ is a $\mathbb{P}$-name such that $\Vdash_{\mathbb{P}} \tau \in 2^\omega \setminus V$. Then for every $p \in \mathbb{P}$ there exists $q \in \mathbb{P}$ such that $q \leq p$ and one of (a), (b) holds.

(a) *$q \Vdash_{\mathbb{P}} V[\tau]$ is a random real extension of V.*

(b) *There is a Borel function $B : 2^\omega \to \omega^\omega$ such that $q \Vdash_{\mathbb{P}} B(\tau)$ is an infinitely often equal real over V.*

For simplicity we assume that p is the trivial condition. For each $n \geq 1$, define $\mathring{T}_n \in V^{\mathbb{Q}_n}$ by

$$\mathring{T}_n = \{\sigma \in {}^{<\omega}2 : (\forall p \in G_{\mathbb{Q}_n})(\exists q \in \mathbb{P})(q \leq p \text{ and } q \Vdash_{\mathbb{P}} \sigma \subseteq \tau)\}$$

Note that $\Vdash_{\mathbb{Q}_n} \mathring{T}_n$ is a leafless subtree of ${}^{<\omega}2$ and

$$\Vdash_{\mathbb{P}} \bigcap_{n \geq 1} [\mathring{T}_n] = \{\tau\}$$

Infinitely often equal or random

Claim

Suppose for some $n \geq 1$ and $q \in \mathbb{Q}_n$, $q \Vdash_{\mathbb{Q}_n} \mathring{T}_n$ is not a perfect tree. Then there exists $p \in \mathbb{P}$ such that $p \leq q$ and $p \Vdash_{\mathbb{P}} V[\tau]$ is a random real extension of V.

Proof: Choose $q_1 \in \mathbb{Q}_n$, $q_1 \leq q$ and $\sigma \in {}^{<\omega}2$ such that $q_1 \Vdash_{\mathbb{Q}_n} \sigma \in \mathring{T}_n$ and $\mathring{T}_n$ has a unique branch above σ. Choose $p \in \mathbb{P}$, $p \leq q_1$ such that $p \Vdash_{\mathbb{P}} \sigma \subseteq \tau$. Let $G_{\mathbb{P}}$ be $\mathbb{P}$-generic over V with $p \in G_{\mathbb{P}}$. Put $G_{\mathbb{Q}_n} = G_{\mathbb{P}} \cap \mathbb{Q}_n$. Then, $\tau[G_{\mathbb{P}}] \in V[G_{\mathbb{Q}_n}]$ since it is the unique branch through $\mathring{T}_n[G_{\mathbb{Q}_n}]$ above σ. Since intermediate models in a random real extension are also random real extensions (as a complete subalgebra of a measure algebra is also a measure algebra), the claim follows. ☕

Infinitely often equal or random

In the other case, we get an infinitely often equal real.

Claim

Suppose for every $n \geq 1$, $\Vdash_{\mathbb{Q}_n} \mathring{T}_n$ is a perfect tree. Then for some Borel function B, $\Vdash_{\mathbb{P}} B(\tau) \in \omega^\omega$ is an infinitely often equal real over V.

We omit the technical details.

A non-meager set in $V^{\mathbb{P}}$

Lemma

Suppose $V \models non(Meager) = \kappa$. Then $V^{\mathbb{P}} \models non(Meager) \leq \kappa$.

Let us recall how Pawlikowski did this for $\mathbb{P} =$ Random. Let $C(2^\omega)$ be the set of all continuous function from 2^ω to 2^ω. It is a Polish space under the topology of uniform convergence. Fix a non-meager subset $\mathcal{F} \subseteq C(2^\omega)$ such that $|\mathcal{F}| = \text{non(Meager)}$. Let $r \in 2^\omega$ be random over V. Then one can show that $X = \{f(r) : f \in \mathcal{F}\}$ is non-meager in $V[r]$. We take a more "combinatorial approach".

A non-meager set in $V^{\mathbb{P}}$

Let $\mathbf{A}$ be the set of all quadruples $\mathbf{x} = (\mathbf{m}, \mathbf{k}, \mathbf{n}, \mathbf{h})$ where

(a) $\mathbf{m} = \langle m_i : i < \omega \rangle$, $\mathbf{n} = \langle n_i : i < \omega \rangle$ and $\mathbf{k} = \langle k_i : i < \omega \rangle$ are in ${}^{\omega}\omega$, $m_0 = 0$ and m_i's are strictly increasing.

(b) $\mathbf{h} = \langle h_i : i < \omega \rangle$ where each $h_i : {}^{k_i}2 \to {}^{[m_i, m_{i+1})}2$.

Let $\{\mathbf{x}_\alpha = (\mathbf{m}_\alpha, \mathbf{n}_\alpha, \mathbf{k}_\alpha, \mathbf{h}_\alpha) : \alpha < \kappa\}$ be non-meager in $\mathbf{A}$ w.r.t. the (Polish) topology generated by declaring finite restrictions of members of $\mathbf{A}$ clopen. Let $\mathring{r}_n \in 2^\omega \cap V^{\mathbb{P}}$ be the random real added by $\mathbb{Q}_n$.

Claim

For each $\alpha < \kappa$, define $\mathring{y}_\alpha \in 2^\omega \cap V^{\mathbb{P}}$ by

$$\mathring{y}_\alpha \restriction [m_{\alpha,i}, m_{\alpha,i+1}) = h_{\alpha,i}(\mathring{r}_{n_{\alpha,i}} \restriction 2^{k_{\alpha,i}})$$

Then $V^{\mathbb{P}} \models \{\mathring{y}_\alpha : \alpha < \kappa\}$ is non-meager.

Generic ultrapowers

Proof of Theorem: Let $\mathbb{P}$, $\langle \mathbb{Q}_n : n \geq 1 \rangle$ and $\langle p_n : n \geq 1 \rangle$ satisfy clauses **(1)-(4)** above. Towards a contradiction, fix a σ-ideal $\mathcal{I}$ on X such that $\mathcal{I}$ is exactly κ-complete and $\mathcal{P}(X)/\mathcal{I}$ is isomorphic to $\mathbb{P}$. Let G be $\mathcal{P}(X)/\mathcal{I}$-generic over V and let $j : V \to N \subseteq V[G]$ be the corresponding generic ultrapower embedding with critical point κ.

Fix $f : X \to \kappa$ in V such that $[f]_G$ represents $\kappa \in N$. Define $\mathcal{J} = \{A \subseteq \kappa : f^{-1}[A] \in \mathcal{I}\}$. Let $\mathbb{Q} = \mathcal{P}(\kappa)/\mathcal{J}$. Then $\mathbb{Q} \lessdot \mathcal{P}(X)/\mathcal{I}$. By a previous lemma, $\mathbb{Q}$ adds a new real. Let H be $\mathbb{Q}$-generic over V and let $k : V \to M \subseteq V[H]$ be the corresponding generic ultrapower embedding with critical point κ. Note that ${}^{\kappa}N \cap V[G] \subseteq N$ and ${}^{\kappa}M \cap V[H] \subseteq M$ and we'll use this freely.

Generic ultrapowers

We have the following two cases.

Case 1: There is a real $r \in {}^{\omega}\omega \cap V[H]$ such that r is infinitely often equal to every real in V. Clearly $r \in M$. Let $\langle r_\alpha : \alpha < \kappa \rangle$ represent r. Then for every $x \in {}^{\omega}\omega \cap V$, there exists $\alpha < \kappa$ such that r_α and x agree infinitely often. It follows that there is a non-meager set of size κ in V. Since $\mathbb{P}$ adds a random real, $V \cap 2^\omega$ is meager in $V[G]$. Let B be a meager F_σ-set coded in $V[G]$ that contains $V \cap 2^\omega$. Then B is also coded in N. So by elementarity of j, it follows that every set of reals in V of size $< \kappa$ is meager in V. So $V \models \text{non(Meager)} = \kappa$. Hence $N \models \text{non(Meager)} = j(\kappa) > \kappa$. By a previous lemma, $V[G]$ and hence N has a non-meager set of size κ: A contradiction.

Generic ultrapowers

Case 2: No real in M is infinitely often equal to every real in V. By a previous lemma, for every new real $r \in M$, $V[r]$ is a random real extension of V. Choose $r \in V[H]$ random over V. Then $r \in M$. Let $\langle r_\alpha : \alpha < \kappa \rangle$ represent r. Then $\{r_\alpha : \alpha < \kappa\}$ is a non-null set in V. Since $\mathbb{P}$ adds a Cohen real, $V \cap 2^\omega$ is null in $V[G]$. Let B be a null G_δ-set coded in $V[G]$ that contains $V \cap 2^\omega$. Then B is also coded in N. So by elementarity of j, it follows that every set of reals in V of size $< \kappa$ is null in V. In particular, for every $\gamma < \kappa$, $\{r_\alpha : \alpha < \gamma\}$ is null in V. By considering $k(\langle r_\alpha : \alpha < \kappa \rangle)$, it follows that $\{r_\alpha : \alpha < \kappa\}$ is null in M. Let $r' \in M$ be the code of a null G_δ-set witnessing this. It follows that $V[r']$ is not a random real extension of V: A contradiction. ☕

Large free sets

For a function $F : X \to [X]^{<\omega}$, we say that $Y \subseteq X$ is F-free iff

$$(\forall x, y \in Y)(x \neq y \implies y \notin F(x))$$

Komjáth's question can be reformulated as follows.

Question

Suppose $X \subseteq [0,1]$ and $F : X \to [X]^{<\omega}$. Must there exist an F-free $Y \subseteq X$ such that $\mu^\star(Y) = \mu^\star(X)$?

Question

Suppose $X \subseteq [0,1]$ is non-null and $f : X \to X$. Must there exist $Y \subseteq X$ such that $\mu^\star(Y) > 0$ and $\mu^\star(X \setminus f^{-1}[Y]) = \mu^\star(X)$?

The transversal theorem says that if f is countable-to-one, then the answer is yes.

Category

Fact

Suppose $X \subseteq [0,1]$, $1 \leq n < \omega$ and $F : X \to [X]^{\leq n}$. Then there exists an F-free $Y \subseteq X$ such that Y is everywhere non-meager in X.

Our proof of this exploits the following. If an atomless forcing $\mathbb{P}$ has a countable dense set subset, then $\mathbb{P}$ is isomorphic to Cohen forcing. For details, see [2].

References

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