Some Ramsey-type problems about measure and category

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Euclidean Ramsey theory

Suppose $\mathcal{F}$ is a family of subsets of $\mathbb{R}^n$ and $X \subseteq \mathbb{R}^n$. We say that X avoids $\mathcal{F}$ iff no set in $\mathcal{F}$ is contained in X.
The chromatic number of $\mathcal{F}$, denoted $\mathrm{Chr}(\mathcal{F})$, is the smallest cardinality of a partition $\mathcal{P}$ of $\mathbb{R}^n$ such that every member of $\mathcal{P}$ avoids $\mathcal{F}$.

- (de Grey, 2018) Let $\mathcal{F} = \{\{x, y\} \subseteq \mathbb{R}^2 : ||x - y|| = 1\}$. Then $5 \leq \mathrm{Chr}(\mathcal{F}) \leq 7$.
- (Gallai, 1939) If $\mathcal{F}$ is the family of all similar copies of a finite subset of $\mathbb{R}^n$, then $\mathrm{Chr}(\mathcal{F}) \geq \omega$.
- (A. Miller, 1989) If $\mathcal{F}$ is the family of all similar copies of an infinite subset of $\mathbb{R}$, then $\mathrm{Chr}(\mathcal{F}) = 2$.

Borel sets

Given a family $\mathcal{F}$ of subsets of $\mathbb{R}^n$ and $X \subseteq \mathbb{R}^n$ which is "large" in some sense, we can ask if there is a "large" subset $Y \subseteq X$ such that Y avoids $\mathcal{F}$. The following fact imposes some natural restrictions.

Exercise

Let $\mathcal{F}$ be a family of subsets of $\mathbb{R}^n$ such that $\mathcal{F}$ is closed under isometries and for some $n < \omega$,

$$\inf \{diam(A) : A \in \mathcal{F}, |A| = n\} = 0$$

Suppose $B \subseteq \mathbb{R}^n$ is Borel and B avoids $\mathcal{F}$. Then B is both meager and Lebesgue null.

Similar copies

Fact

Suppose $A \subseteq \mathbb{R}^n$ is bounded and countable, $B \subseteq \mathbb{R}^n$ is Borel and no similar copy of A is contained in B. Then B is meager.

Question (Erdős, 1974)

Suppose $A \subseteq \mathbb{R}$ is countably infinite. Is there a compact $K \subseteq \mathbb{R}$ of positive measure such that no similar copy of A is contained in K?

The answer is positive if

- (Eigen and Falconer, 1985) $A = \{n^{-1} : n \geq 1\}$.
- (Bourgain, 1987) $A = X + X + X$ for some infinite X.

Erdös' problem remains open for $A = \{2^{-n} : n \geq 1\}$.

Large sets in $\mathbb{R}^n$: Measure

(1) For $X \subseteq \mathbb{R}^n$, $\mu_n^\star(X)$ denotes the n-dimensional Lebesgue outer measure of X. X is (Lebesgue) null iff $\mu_n^\star(X) = 0$. If the dimension is clear from the context, we drop the n and write $\mu^\star$.

(2) Suppose $Y \subseteq X \subseteq \mathbb{R}^n$. We say that Y has full outer measure in X iff for every compact $K \subseteq \mathbb{R}^n$, $\mu^\star(X \cap K) = \mu^\star(Y \cap K)$. We say that Y has zero inner measure in X iff $X \setminus Y$ has full outer measure in X. Otherwise, Y has positive inner measure in X.

Large sets in $\mathbb{R}^n$: Category

Suppose $Y \subseteq X \subseteq \mathbb{R}^n$. We say that Y is everywhere nonmeager in X iff for every open $U \subseteq \mathbb{R}^n$, if $X \cap U$ is nonmeager, then $Y \cap U$ is nonmeager.

Independence results

Let

$$\mathcal{F} = \{\{a, b, c\} \subseteq \mathbb{R}^2 : a, b, c \text{ form the vertices of a right triangle}\}$$

Erdős and Komjáth (1990) showed that $\mathrm{Chr}(\mathcal{F}) = \omega$ iff CH holds.

Theorem (Komjáth, 1991)

It is consistent that there is a non-meager subset of plane each of whose non-meager subsets contains the vertices of a right triangle.

Theorem (Shelah, 2000)

It is consistent that there is a non-null subset of plane each of whose non-null subsets contains the vertices of a right triangle.

Rational distances

Let $\mathcal{F}_n = \{\{x, y\} \subseteq \mathbb{R}^n : 0 < ||x - y|| \in \mathbb{Q}\}$. Erdős and Hajnal (1969) showed that $\mathrm{Chr}(\mathcal{F}_2) = \omega$. Komjáth (1994) showed that for every n, $\mathrm{Chr}(\mathcal{F}_n) = \omega$.

Question (Komjáth, 1994)

Suppose $X \subseteq \mathbb{R}^n$. Must there exist $Y \subseteq X$ such that Y has full outer measure in X and Y avoids $\mathcal{F}_n$? Must there exist an everywhere nonmeager subset $Y \subseteq X$ such that Y avoids $\mathcal{F}_n$?

Note that under CH/MA, the answer is yes in all dimensions. But we think that ZFC suffices.

Theorem (Kumar, 2012)

Yes to both, if $n = 1$.

The proof uses a result of Gitik and Shelah which says that forcing with a σ-ideal cannot be isomorphic to the either one of Cohen and Random $\times$ Cohen.

In plane

Theorem (Erdős-Hajnal, 1969)

There is a well-ordering $(\mathbb{R}^2, \prec)$ *such that for every* $x \in \mathbb{R}^2$*,* $\{y : y \prec x, ||x - y|| \in \mathbb{Q}\}$ *is finite.*

The coloring number of a graph $G = (V, E)$ is the least cardinal κ such that there is a well-order $\prec$ of V such that for every $x \in V$, $|\{y : y \prec x, \{x, y\} \in E\}| < \kappa$.

Question (Komjáth, 2016)

Suppose $X \subseteq [0, 1]$ *and* (X, E) *is a graph of countable coloring number. Must there exist an* E*-independent* $Y \subseteq X$ *such that* Y *has full outer measure in* X*? Must there exist an* E*-independent* $Y \subseteq X$ *such that* Y *is everywhere nonmeager in* X*?*

Some partial results

Suppose $X \subseteq [0,1]$ and (X, E) is a graph of coloring number $\kappa \leq \omega$.

1. If $\kappa < \omega$, then there exists an E-independent $Y \subseteq X$ such that Y is everywhere nonmeager in X.
2. If $\mathrm{Null} \restriction X$ is nowhere ω_1-saturated, then there exists an E-independent $Y \subseteq X$ such that Y has full outer measure in X. Similarly for category.

A transversal of full outer measure

Theorem (Kumar-Shelah, 2017)

Suppose $\{X_\alpha : \alpha \in S\}$ *is a partition of* $X \subseteq [0,1]$ *into countable sets. Then there exists* $Y \subseteq X$ *such that* $\mu^\star(Y) = \mu^\star(X)$ *and* $|Y \cap X_\alpha| = 1$ *for each* $\alpha \in S$.

Lemma (Baby transversal)

Suppose $\{X_\alpha : \alpha \in S\}$ *is a partition of* $X \subseteq [0,1]$ *into finite sets. Then there exists* $Y \subseteq X$ *such that* $\mu^\star(Y) = \mu^\star(X)$ *and* $|Y \cap X_\alpha| = 1$ *for each* $\alpha \in S$.

Forcing

Theorem (Kumar-Shelah, 2017)

Let $\mathbb{P}$ be a forcing. Suppose there are $\langle \mathbb{Q}_n : n \geq 1 \rangle$ and $\langle p_n : n \geq 1 \rangle$ satisfying (1)-(4) below. Then forcing with a σ-ideal cannot be isomorphic to $\mathbb{P}$.

(1) *For every $n \geq 1$, $p_n \in \mathbb{P}_n$ and $\mathbb{Q}_n \lessdot \mathbb{P}_{\leq p_n}$.*

(2) *$\bigcup \{\mathbb{Q}_n : n \geq 1\}$ is dense in $\mathbb{P}$.*

(3) *Each $\mathbb{Q}_n$ is isomorphic to random forcing.*

(4) *$\mathbb{P}$ adds a Cohen real.*

Note that both Random $\times$ Cohen and a finite support iteration of random forcing of length ω are two simple examples of such $\mathbb{P}$'s. Fremlin (see Section 547 in his Measure theory, Vol. 5 Part II) observed that requirement (4) can be dropped by showing that if $\mathbb{P}$ satisfies (1)-(3), then either $\mathbb{P}$ adds a Cohen real or there exists $p \in \mathbb{P}$ such that $\mathbb{P}_{\leq p}$ is isomorphic to Random. The latter case was ruled out by Gitik and Shelah.

ε-progress

Given $X \subseteq [0,1]$, a partition $\{X_\alpha : \alpha \in S\}$ of X where each $|X_\alpha| = \omega$ and enumerations $\{x_{\alpha,n} : n < \omega\}$ of X_α for each $\alpha \in S$, define $Y_n = \{x_{\alpha,n} : \alpha \in S\}$ and $Y_n \restriction W = \{x_{\alpha,n} : \alpha \in W\}$ for $W \subseteq S$. Let $(\star)$ be the following statement.

$(\star)$: For every $X \subseteq [0,1]$, for every partition $\{X_\alpha : \alpha \in S\}$ of X into sets of size ω, for every enumeration $X_\alpha = \{x_{\alpha,n} : n < \omega\}$ (so we can speak of Y_n's w.r.t. this enumeration), there is a subset W of S such that either

(a) Y_0 is null or

(b) $Y_0 \restriction W$ has positive outer measure and for all $n \geq 1$, $Y_n \restriction W$ has zero inner measure in Y_n.

Then $(\star)$ implies that the transversal theorem holds.

Failure of $(\star)$

Assume $(\star)$ fails. Fix a witnessing $X \subseteq [0,1]$, a partition $\{X_\alpha : \alpha \in S\}$, enumerations $X_\alpha = \{x_{\alpha,n} : n \in \omega\}$ and the corresponding Y_n's. By thinning out S, we can assume the following.

1. For each $n < \omega$, Y_n is non-null.
2. If $n < \omega$, $W \subseteq S$ and $Y_0 \restriction W$ is null, then $Y_n \restriction W$ has zero inner measure in Y_n.
3. For each $n \geq 1$, there exists $W_n \subseteq S$ such that $Y_n \restriction W_n$ is null and if $W \subseteq S \setminus W_n$ is such that $Y_n \restriction W$ is null, then for all $m \geq n+1$, $Y_m \restriction W$ has zero inner measure in Y_m. Fix such W_n's.

Let $\mathcal{I} = \{W \subseteq S : \mu(Y_0 \restriction W) = 0\}$ and $\mathbb{P} = \mathcal{I}^+$ be the forcing with $\mathcal{I}$. Observe that $Z \in \mathbb{P}$ iff $(\exists^\infty n)(Y_n \restriction Z$ has positive inner measure in $Y_n)$.

Failure of $(\star)$

For each $n \geq 1$, define $B_n = \mathrm{env}(Y_n)$ and

$$\mathbb{Q}_n = \{Z : (\exists B \subseteq B_n)(\mu(B) > 0 \text{ and } Z = \{\alpha \in S \setminus W_n : x_{\alpha,n} \in B \cap Y_n\})\}$$

Note that if $Z \in \mathbb{Q}_n$, then $Y_n \restriction Z$ has positive inner measure in Y_n and hence $Y_0 \restriction Z$ is non-null. So $\mathbb{Q}_n \subseteq \mathbb{P}$ for every $n \geq 1$. Let $p_n = S \setminus W_n$.

Claim

(1) *For each $n \geq 1$, $\mathbb{Q}_n \lessdot \mathbb{P}_{\leq p_n}$.*

(2) *$\bigcup\{\mathbb{Q}_n : n \geq 1\}$ is dense in $\mathbb{P}$.*

(3) *Each $\mathbb{Q}_n$ is isomorphic to random forcing.*

(4) *$\mathbb{P}$ adds a Cohen real.*

Cohen real

Proof of (4): Construct a tree $\langle A_\sigma : \sigma \in {}^{<\omega}2\rangle$ of subsets of S such that the following hold.

(a) $A_{\langle\rangle} = S$, and for every $\sigma \in {}^{<\omega}2$

(b) A_σ is a disjoint union of $A_{\sigma 0}$ and $A_{\sigma 1}$,

(c) $0 \leq k \leq |\sigma| + 1 \implies \mu^\star(Y_k \restriction A_{\sigma 0}) = \mu^\star(Y_k \restriction A_{\sigma 1}) = \mu^\star(Y_k \restriction A_\sigma)$.

For clause (c), we make use of baby transversal lemma. We claim that if G is $\mathbb{P}$-generic over V, then in $V[G]$, the real $x = \bigcup\{\sigma \in {}^{<\omega}2 : A_\sigma \in G\}$ is Cohen over V. To see this, suppose that $D \subseteq {}^{<\omega}2$ is dense and $x \notin \bigcup\{[\sigma] : \sigma \in D\}$. Then $S \setminus \bigcup\{A_\sigma : \sigma \in D\} = Z \in G$ so that $Y_0 \restriction Z$ is non-null. But then, for some $n \geq 1$, $Y_n \restriction Z$ has positive inner measure in Y_n. So Z meets A_σ for every σ extending some τ, $|\tau| = n$, and hence also meets some condition in $\{A_\sigma : \sigma \in D\}$. A contradiction. □

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