# Forcing with $\sigma$-ideals

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# Forcing with ideals

We say that $\mathbb{P}$ and $\mathbb{Q}$ are forcing isomorphic (and write $\mathbb{P} \cong \mathbb{Q}$) iff their boolean completions are isomorphic.
If $\mathcal{I}$ is an ideal on $X$, forcing with $\mathcal{I}$ refers to either $\mathcal{I}^+$ ordered by inclusion modulo $\mathcal{I}$ or its separative quotient $\mathcal{P}(X)/\mathcal{I}$. Note that every forcing is isomorphic to forcing with some ideal (by Sikorski extension theorem).

## Theorem (Solovay, 1971)

*Forcing with a $\sigma$-ideal can be isomorphic to an atomless measure algebra.*

Fremlin asked if it could be isomorphic to random forcing.

## Theorem (Gitik-Shelah, 1989)

*Forcing with a $\sigma$-ideal cannot be isomorphic to any of the following: Cohen, Random, Sacks, Hechler, Laver, Mathias, Miller.*

# Generalizations?

**Problem (Shelah, 1999)**

*Show that forcing with a a $\sigma$-ideal cannot be isomorphic to a Suslin proper (or just ccc) forcing which is generated by the name of one real.*

# Generic ultrapowers

Let $\mathcal{I}$ be a $\sigma$-ideal on $X$. For $Y \subseteq X$, $Y/\mathcal{I} = \{Z \subseteq X : Y \Delta Z \in \mathcal{I}\}$. Put $\mathbb{P} = \mathcal{P}(X)/\mathcal{I}$. Let $G$ be $\mathbb{P}$-generic over $V$. Working in $V[G]$, define the following.

1. ${}^X V \cap V = \{f \in V : f$ is a function with domain $X\}$.
2. For $f, g \in {}^X V \cap V$, $f \sim_G g$ iff $\{x : f(x) = g(x)\}/\mathcal{I} \in G$ and let $[f]_G$ denote the equivalence class.
3. $({}^X V \cap V)/G = \{[f]_G : f \in {}^X V \cap V\}$
4. $([f]_G, [g]_G) \in E$ iff $\{x \in X : f(x) \in g(x)\}/\mathcal{I} \in G$.

The structure $(({}^X V \cap V)/G, E)$ is the $G$-ultrapower of $V$.

# Precipitous ideals

Let $\mathcal{I}$ be a $\sigma$-ideal on $X$ and $\mathbb{P} = \mathcal{P}(X)/\mathcal{I}$. We say that $\mathcal{I}$ is precipitous iff for every $\mathcal{P}(X)/\mathcal{I}$-generic filter $G$ over $V$, $(({}^X V \cap V)/G, E)$ is well-founded.

## Fact

(a) *$\mathcal{I}$ is precipitous iff for every $A \in \mathcal{I}^+$ and sequence $\langle \mathcal{A}_n : n < \omega \rangle$ of maximal antichains in $\mathcal{I}^+$ below $A$ satisfying $(\forall Z \in \mathcal{A}_{n+1})(\exists Y \in \mathcal{A}_n)(Z \subseteq Y)$, there exists $\langle Y_n : n < \omega \rangle$ such that each $Y_n \in \mathcal{A}_n$ and $\bigcap \{ Y_n : n < \omega \} \neq \emptyset$.*

(b) *If $\mathbb{P}$ is proper, then $\mathcal{I}$ is precipitous.*

Suppose $\mathcal{I}$ is a precipitous and $G$ is $\mathcal{P}(X)/\mathcal{I}$-generic over $V$. Let $M$ be the transitive collapse of the $G$-ultrapower of $V$ and $\pi : (({}^X V \cap V)/G, E) \cong (M, \in)$ the corresponding isomorphism. Define $j : V \to M$ by $j(y) = \pi([c_y])$ where $c_y : X \to \{y\}$. It is easy to check that $j$ is an elementary embedding. $j$ is called the generic ultrapower embedding associated with $G$.

# Critical point

## Definition

*Let $\mathcal{I}$ be an $\sigma$-ideal on $X$. We say that $\mathcal{I}$ is exactly $\kappa$-complete iff for every $A \in \mathcal{I}^+$, add($\mathcal{I}$) $= \kappa$.*

## Exercise

*Let $\mathcal{I}$ be a $\sigma$-ideal on $X$. Then there is a partition $\mathcal{F} \subseteq \mathcal{I}^+$ of $X$ such that for each $Y \in \mathcal{F}$, there exists a regular uncountable $\kappa$ such that $\mathcal{I} \restriction Y = \{A \subseteq X : A \cap Y \in \mathcal{I}\}$ is an exactly $\kappa$-complete ideal on $X$.*

## Fact

*Suppose $\mathcal{I}$ is a precipitous $\sigma$-ideal on $X$, $G$ is $\mathcal{P}(X)/\mathcal{I}$-generic over $V$ and $j : V \to M$ is the associated generic ultrapower embedding. Let $\kappa = \max(\{add(\mathcal{I} \restriction Y) : Y/\mathcal{I} \in G\})$ (It exists by previous exercise). Then $j(\kappa) > \kappa$ and $j \restriction \kappa$ is the identity. We call $\kappa$ the critical point of $j$.*

# Closure of ultrapowers

Let $\mathcal{I}$ be a $\sigma$-ideal on $X$ and $\mathbb{P} = \mathcal{P}(X)/\mathcal{I}$. We say that $\mathcal{I}$ has the disjointing property iff for every antichain $\mathcal{A} \subseteq \mathbb{P}$, there exists $f : \mathcal{A} \to \mathcal{P}(X)$ such that for every $a \in \mathcal{A}$, $a = f(a)/\mathcal{I}$ and $\langle f(a) : a \in \mathcal{A}\rangle$ has pairwise disjoint sets. Note that if $\mathbb{P}$ has the ccc (equivalently, $\mathcal{I}$ is $\omega_1$-saturated), then $\mathcal{I}$ has the disjointing property.

## Fact

*Suppose $\mathcal{I}$ is a $\sigma$-ideal on $X$ that has the disjointing property. Then $\mathcal{I}$ is precipitous. Let $G$ be $\mathcal{P}(X)/\mathcal{I}$-generic over $V$ and $j : V \to M$, the associated generic ultrapower embedding with critical point $\kappa$. Then ${}^{\kappa}M \cap V[G] \subseteq M$.*

# Hechler forcing

Let $\mathbb{H}$ be the Hechler forcing: $p \in \mathbb{H}$ iff $p = (\sigma_p, f_p)$ where $\sigma_p \in \omega^{<\omega}$, $f_p \in \omega^\omega$ is strictly increasing and $\sigma_p \subseteq f_p$. For $p, q \in \mathbb{H}$, $p \leq q$ iff $\sigma_q \subseteq \sigma_p$ and $(\forall n < \omega)(f_q(n) \leq f_p(n))$. Note that $\mathbb{H}$ is $\sigma$-centered.

## Theorem (Gitik-Shelah, 1993)

*Suppose $\mathcal{I}$ is a $\sigma$-ideal on $X$ and $\mathbb{P} = \mathcal{P}(X)/\mathcal{I}$. Then $\mathbb{H}$ cannot be isomorphic to $\mathbb{P}$.*

**Proof**. Suppose not. By restricting $\mathcal{I}$ to some $Y \in \mathcal{I}^+$, we can assume that $\mathcal{I}$ is exactly $\kappa$-complete. Since $\mathbb{H}$ is ccc, $\mathcal{I}$ is an $\omega_1$-saturated ideal on $X$. So $\kappa$ is weakly inaccessible (Ulam matrix). Let $G$ be $\mathbb{P}$-generic over $V$ and $j : V \to M$, the associated generic ultrapower embedding with critical point $\kappa$. Note that ${}^\kappa M \cap V[G] \subseteq M$. Brendle, Judah and Shelah showed that $V^{\mathbb{H}} \models \mathfrak{b} = \omega_1$. So $V[G]$ and therefore $M$ thinks that there is an unbounded family of size $\omega_1$. By elementarity of $j$, for some $Y = \{y_i : i < \omega_1\} \subseteq \omega^\omega$, $V \models Y$ is an unbounded family. Since $M$ contains a dominating real over $V$, $M \models j(Y) = Y$ is bounded. Hence $V \models Y$ is bounded: Contradiction. ☕

# Eventually different real forcing

Let $\mathbb{E}$ be the eventually different real forcing defined as follows.

(1) $p \in \mathbb{E}$ iff $p = (\sigma_p, F_p)$ where $\sigma_p \in \omega^{<\omega}$ and $F_p \subseteq \omega^\omega$ is finite.

(2) $p \leq q$ iff $\sigma_q \subseteq \sigma_p$, $F_q \subseteq F_p$ and $(\forall n \in \mathrm{dom}(\sigma_p) \setminus \mathrm{dom}(\sigma_q))(\forall x \in F_q)(\sigma_p(n) \neq x(n))$.

**Fact (Brendle, 1995)**

*$V^{\mathbb{E}} \models non(Meager) = \omega_1$. In fact, $\mathbb{E}$ adds a Lusin set of size continuum.*

**Exercise**

*Suppose $\mathcal{I}$ is a $\sigma$-ideal on $X$ and $\mathbb{P} = \mathcal{P}(X)/\mathcal{I}$. Then $\mathbb{E}$ cannot be isomorphic to $\mathbb{P}$.*

# Product of Cohen and random

Let $\mathsf{Random}_\theta$ (resp. $\mathsf{Cohen}_\theta$) denote the forcing for adding $\theta$ random (resp. Cohen) reals. $\mathsf{Random}_\theta$ is the measure algebra on the product measure space $[0,1]^\theta$ (with Lebesgue measure on $[0,1]$) and $\mathsf{Cohen}_\theta$ is the forcing for adding a function in ${}^{\omega\times\theta}2$ using finite approximations.

## Fact (Solovay, 1971)

*Suppose $\kappa$ is measurable with a witnessing normal prime ideal $\mathcal{I}$ and $\lambda = 2^\kappa$. Let $\mathcal{J}$ be the ideal generated by $\mathcal{I}$ in $V^{\mathsf{Random}_\kappa}$. Then $V^{\mathsf{Random}_\kappa} \models \mathcal{P}(\kappa)/\mathcal{J} \cong \mathsf{Random}_\lambda$.*

## Fact (Kunen)

*Suppose $\kappa$ is measurable with a witnessing normal prime ideal $\mathcal{I}$ and $\lambda = 2^\kappa$. Let $\mathcal{J}$ be the ideal generated by $\mathcal{I}$ in $V^{\mathsf{Cohen}_\kappa}$. Then $V^{\mathsf{Cohen}_\kappa} \models \mathcal{P}(\kappa)/\mathcal{J} \cong \mathsf{Cohen}_\lambda$.*

# Product of Cohen and random

Gitik and Shelah showed that forcing with a $\sigma$-ideal cannot be isomorphic to Random $\times$ Cohen. Fremlin asked if it could be isomorphic to $\text{Random}_{\theta_1} \times \text{Cohen}_{\theta_2}$ for some $\theta_1, \theta_2 > 0$.

## Theorem (Kumar-Shelah, 2017)

*Suppose $\mathcal{I}$ is a $\sigma$-ideal on $X$ and $\mathbb{P} = \mathcal{P}(X)/\mathcal{I}$. Let $\theta_1, \theta_2$ be positive cardinals. Then $\text{Random}_{\theta_1} \times \text{Cohen}_{\theta_2}$ cannot be isomorphic to $\mathbb{P}$.*

**Proof**. Suppose not. By restricting $\mathcal{I}$, we can assume $\text{add}(\mathcal{I} \restriction A) = \kappa$ for every $A \in \mathcal{I}^+$. Note that since $\mathbb{P}$ is ccc, $\kappa$ is weakly inaccessible. Let $G$ be $\mathbb{P}$-generic over $V$ and let $j : V \to M \subseteq V[G]$ be the associated generic ultrapower embedding with critical point $\kappa$. The following is well known: $V^{\text{Cohen}_1} \models \text{cov}(\text{Null}) = \omega_1$. So $V[G]$ and therefore $M$ (as ${}^{\kappa}M \cap V[G] \subseteq M$) thinks that $\text{cov}(\text{Null}) = \omega_1$. Hence also $V \models \text{cov}(\text{Null}) = \omega_1$. Let $\overline{A} = \langle A_i : i < \omega_1 \rangle$ be a sequence of null $G_\delta$-sets witnessing this in $V$. Note that $j(\overline{A}) = \overline{A}$. By elementarity of $j$, $M \models \bigcup\{A_i : i < \omega_1\} = \mathbb{R}$. But this is impossible as $V[G]$ and therefore $M$ contains a random real over $V$ which cannot belong to any of the $A_i$'s. ☕

# Lusin's theorem

Let $\mu^\star$ denote Lebesgue outer measure in $\mathbb{R}^n$ and $\mu$ be its restriction to Lebesgue measurable sets. For $X \subseteq \mathbb{R}^n$, let $\mathrm{env}(X)$ be a $G_\delta$-envelope of $X$. This means that $X \subseteq \mathrm{env}(X)$ and every compact subset of $\mathrm{env}(X) \setminus X$ is Lebesgue null.

## Theorem (Lusin, 1934)

*For every $X \subseteq \mathbb{R}^n$, there exists $X' \subseteq X$ such that $\mu^\star(X') = \mu^\star(X \setminus X') = \mu^\star(X)$. Here $\mu^\star$ denotes the n-dimensional Lebesgue outer measure.*

**Proof**. It suffices to show the following. Whenever $\mu^\star(X) > 0$, there exists $Y \subseteq X$ such that $\mu^\star(Y) > 0$ and $\mu^\star(X) = \mu^\star(X \setminus Y)$. For then we can take $X'$ to be the (countable) union of a maximal family of such $Y$'s with pairwise disjoint envelopes. Suppose this fails for some $X$. By thinning out $X$, we can assume that for some regular $\kappa$, $\mathrm{add}(\mathrm{Null} \restriction A) = \kappa$ for every non-null $A \subseteq X$. Put $\mathbb{P} = \mathcal{P}(X)/\mathrm{Null}$.

# Lusin's theorem

Note that $\mathbb{P} \cong$ Random via the map $A/\text{Null} \mapsto \text{env}(A)$. The main point here is the following: If $A_1 \cap A_2 = \emptyset$, then $E = \text{env}(A_1) \cap \text{env}(A_2)$ is null. Since if $\mu(E) > 0$, then letting $Y = A_1 \cap E$, we have $\mu^\star(Y) = \mu(E) > 0$ and $\mu^\star(X \setminus Y) = \mu^\star(X)$ which is impossible.

Let $G$ be $\mathbb{P}$-generic over $V$ and $j : V \to M \subseteq V[G]$ the associated generic ultrapower embedding with critical point $\kappa$. Note that $\kappa$ is weakly inaccessible and $M$ is closed under $\kappa$-sequences in $V[G]$. Fix $\langle B_i : i < \kappa \rangle$ such that $B_i$'s are increasing null sets with union $X$. Suppose $j(\langle B_i : i < \kappa \rangle) = \langle B'_i : i < j(\kappa) \rangle$. Then for each $i < \kappa$, $B'_i = j(B_i) \supseteq B_i$. Now $M \models B'_\kappa$ is null. Hence $V[G] \models X \subseteq B'_\kappa$ is null. But $V[G]$ is a random real extension of $V$ and $X$ is non-null in $V$: Contradiction. ☕

# Baby transversal

**Lemma (Baby transversal)**

*Suppose* $\{X_\alpha : \alpha \in S\}$ *is a partition of* $X \subseteq [0,1]$ *into finite sets. Then there exists* $Y \subseteq X$ *such that* $\mu^\star(Y) = \mu^\star(X)$ *and* $|Y \cap X_\alpha| = 1$ *for each* $\alpha \in S$*.*

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