On convex bi-embeddability between countable linear orders

(joint work with Alberto Marcone and Luca Motto Ros)

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Introduction

A **classification problem** consists of a pair (X, E), where X is a set and E an equivalence relation on X. A solution is given by an assignment of complete invariants to elements of X.

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A tool from descriptive set theory used to solve and compare some classification problems is the **Borel reducibility**.

- Let E and F be two equivalence relations on the standard Borel spaces X and Y. Then $E \leq_B F$ iff there exists a Borel map $\varphi : X \to Y$ such that

$$x \; E \; y \iff \varphi(x) \; F \; \varphi(y),$$

  for all $x, y \in X$.

- We say that E and F are Borel bi-reducible, and write $E \sim_B F$, if $E \leq_B F$ and $F \leq_B E$.

Let X be a Polish or a standard Borel space. A subset $A \subseteq X$ is **analytic** (or $\mathbf{\Sigma}^1_1$) if there is a Borel subset C of $X \times \omega^\omega$ such that for all $x \in X$,

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i.e. A is the projection on the first coordinate of C.

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Example

Let LO be the Polish space of codes for linear orders on ω, i.e.

$$LO = \{L \in 2^{\omega \times \omega} : L \text{ codes a linear order}\},$$

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Hence, the complexity of $\cong_{LO}$ is very high.

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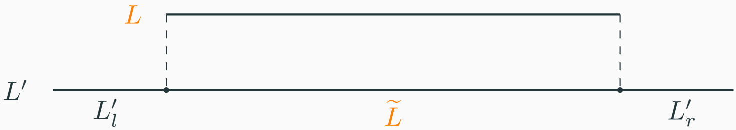
Let LO be the Polish space of codes for linear orders on ω. Consider the relation of **convex embeddability** $\trianglelefteq_{LO}$ between two linear orders L and L' (R. Bonnet, E. Corominas and M. Pouzet, 1973):

$$L \trianglelefteq L' \text{ if } L \text{ is isomorphic to an } L'\text{-interval.}$$



Combinatorial properties and complexity

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- One can embed $(LO, \trianglelefteq_{\mathrm{LO}})$ in $(\triangledown_L, \trianglelefteq_{\mathrm{LO}})$ for every $L \in LO$, where $\triangledown_L = \{L' \in LO \mid L \trianglelefteq_{\mathrm{LO}} L'\}$.

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- $\trianglelefteq_{LO}$ is a proper analytic quasi-order.
- $\trianglelefteq_{LO}$ is not complete for analytic quasi-orders.

Convex bi-embeddability on LO

We call **convex bi-embeddability**, and denote by $\bowtie_{LO}$, the equivalence relation on LO induced by $\trianglelefteq_{LO}$.

Clearly, for $L, L' \in LO$,

$$L \cong_{LO} L' \Rightarrow L \bowtie_{LO} L',$$

but the converse is not true.

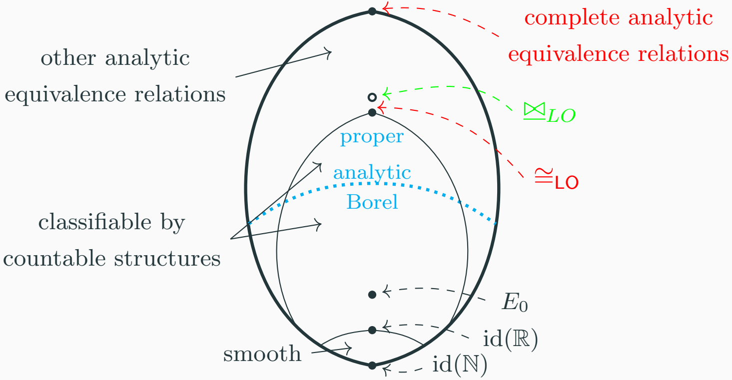
Example

$\omega + \mathbb{Z}\omega \bowtie_{\mathrm{LO}} \mathbb{Z}\omega$, but $\omega + \mathbb{Z}\omega \ncong_{LO} \mathbb{Z}\omega$.

Complexity with respect to Borel Reducibility

Theorem [I. - Motto Ros]

1. $\cong_{LO} \leq_B \bowtie_{LO}$.
2. $\bowtie_{LO}$ is not complete for analytic equivalence relations, i.e. there is an analytic equivalence relation F such that $F \nleq_B \bowtie_{LO}$.



We have evidence that $\bowtie_{LO}$ is not much more complicated than $\cong_{LO}$.

Indeed, we identified a partition of LO that consists of four subsets such that the restriction of $\bowtie_{\mathrm{LO}}$ on each of them is Borel bi-reducible to $\cong_{\mathrm{LO}}$.

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Definition

Let L be a linear order. We say that

- L is **right compressible** if we can write it as $L = L' + L_r$, with $L' \cong L$ and $L_r \neq \varnothing$;
- L is **left compressible** if $L = L_l + L'$, with $L' \cong L$ and $L_l \neq \varnothing$;
- L is **bicompressible** if $L = L_l + L' + L_r$, with $L' \cong L$, $L_l \neq \varnothing$ and $L_r \neq \varnothing$ ($\iff$ L is both left and right compressible);
- L is **incompressible** if it is neither left nor right-compressible.

As examples of countable linear orders with these properties we have ω, ω^*, $\mathbb{Q}$ and $\mathbb{Z}$.

We denote by $LO_r \subseteq LO$ ($LO_l \subseteq LO$) the set of (codes for) countable right-compressible (left-compressible) linear orders.

$LO \setminus (LO_r \cup LO_l)$ $\mathbb{Z}$	$LO_r \setminus LO_l$ ω^*
$LO_l \setminus LO_r$ ω	$LO_r \cap LO_l$ $\mathbb{Q}$

Theorem [I. - Motto Ros]

(i) $\bowtie_{LO} \restriction$ (just) left-compressible $\sim_B \cong_{LO} \restriction$ (just) left-compressible.

(ii) $\bowtie_{LO} \restriction$ (just) right-compressible $\sim_B$ $\cong_{LO} \restriction$ (just) right-compressible.

(iii) $\bowtie_{LO} \restriction$ bi-compressible $\sim_B \cong_{LO} \restriction$ bi-compressible.

(iv) $\bowtie_{LO} \restriction$ incompressible $= \cong_{LO} \restriction$ incompressible.

Given a subset A of a Polish space (or standard Borel space) X, recall that:

- A is **co-analytic** (or $\boldsymbol{\Pi}^1_1$) if $X \setminus A$ is analytic;
- A is $D_2(\boldsymbol{\Pi^1_1})$ if it is the intersection between an analytic set and a co-analytic set.

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Theorem [I. - Marcone]

(i) $LO \setminus (LO_l \cup LO_r)$ is $\mathbf{\Pi}^1_1$-complete.

(ii) $LO_r \cap LO_l$ is $\mathbf{\Sigma}^1_1$-complete.

(iii) $LO_r \setminus LO_l$ is $D_2(\mathbf{\Pi}^1_1)$-complete.

(iv) $LO_l \setminus LO_r$ is $D_2(\mathbf{\Pi}^1_1)$-complete.

Some Applications

Arcs and their classification

Definition

Let $\bar{B}$ be a space homeomorphic to a closed ball in $\mathbb{R}^3$. Given a map $f\colon [0;1] \to \bar{B}$, we say that the pair $(\bar{B}, \operatorname{Im} f)$ is a **proper arc** in $\bar{B}$ if f is a topological embedding and $f(x) \in \partial\bar{B} \iff x = 0$ or $x = 1$. The collection of proper arcs is denoted by $\mathcal{A}$.

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Definition

Two proper arcs $(\bar{B}, f)$ and $(\bar{B}', f')$ are **equivalent**, in symbols

$$(\bar{B}, f) \equiv_{\mathcal{A}} (\bar{B}', f'),$$

if there exists a homeomorphism $\varphi : \bar{B} \to \bar{B}'$ such that $\varphi(f) = f'$.

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The relation $\equiv_{\mathcal{A}}$ is an analytic equivalence relation on $\mathcal{A}$.

Theorem (V. Kulikov, 2017)

$\cong_{LO} \leq_B \equiv_{\mathcal{A}}$.

Definition

Let $(\bar{B}, f), (\bar{B}', g) \in \mathcal{A}$. We say that $(\bar{B}', g)$ is a **component** of $(\bar{B}, f)$, and set

$$(\bar{B}', g) \precsim_{\mathcal{A}} (\bar{B}, f),$$

if there exists a topological embedding $\varphi : \bar{B}' \to \bar{B}$ such that $\varphi(g) = f \cap \mathrm{Im}\ \varphi$.

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The analytic equivalence relation associated to $\precsim_{\mathcal{A}}$ is denoted by $\approx_{\mathcal{A}}$.

Theorem [I. - Motto Ros]

$\trianglelefteq_{LO} \leq_B \precsim_{\mathcal{A}}$, thus also $\bowtie_{LO} \leq_B \approx_{\mathcal{A}}$.

Corollary

(a) The quasi-order $\precsim_{\mathcal{A}}$ has chains and antichains of size the continuum, and thus it is not a wqo.

(b) $\cong_{LO} \leq_B \approx_{\mathcal{A}}$.

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(b) $\cong_{LO} \leq_B \approx_{\mathcal{A}}$.

- We don't know if $\approx_{\mathcal{A}}$ is complete for analytic equivalence relations.
- The previous theorem can be seen as a variation of Kulikov's result.

$$\begin{array}{ccc} \cong_{LO} & \leq_B & \equiv_{\mathcal{A}} \\ \Downarrow & & \Downarrow \\ \bowtie_{LO} & \leq_B & \approx_{\mathcal{A}} \end{array}$$

Definition (E. Glasner - M. Megrelishvili, 2019)

A ternary relation $C \subset X^3$ on a set X is said to be a **circular order** if the following conditions are satisfied:

- Cyclicity: $(x, y, z) \in C \Rightarrow (y, z, x) \in C$;
- Asymmetry: $(x, y, z) \in C \Rightarrow (y, x, z) \notin C$;
- Transitivity: $(x, y, z), (x, z, w) \in C \Rightarrow (x, y, w) \in C$;
- Totality: if $x, y, z \in X$ are distinct, then $(x, y, z) \in C$ or $(x, z, y) \in C$.

Denote by CO the Polish space of codes for circular orders on ω, i.e.

$$CO = \{C \in 2^{\omega \times \omega \times \omega} : C \text{ codes a circular order}\}.$$

The isomorphism relation on CO

Definition

Let $C, C' \in CO$. We say that C and C' are **circularly isomorphic**, and write $C \cong_{CO} C'$, if there exists a bijective function between them which preserves the circular order.

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Every $L \in LO$ defines a standard circular order $C_L \in CO$ as follows:

$C_L(n, m, k) \iff (n <_L m <_L k) \vee (m <_L k <_L n) \vee (k <_L n <_L m)$.

Clearly, for $L, L' \in LO$,

$$L \cong_{LO} L' \Rightarrow C_L \cong_{CO} C_{L'}.$$

Example

$\omega + 1 \ncong_{LO} \omega$, but $C_{\omega+1} \cong_{CO} C_{\omega}$.

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Theorem [I. - Marcone]

$\cong_{CO} \sim_B \cong_{LO}$.

Convex embeddability on CO

Definition (B. Kulpeshov, H. D. Macpherson, 2005)

Let $A \subseteq C$, where C is a circular order. The set A is said to be **convex** in C if for any $x, y \in A$ one of the following holds:

1. for any $z \in C$ with $C(x, z, y)$ we have $z \in A$;
2. for any $z \in C$ with $C(y, z, x)$ we have $z \in A$.

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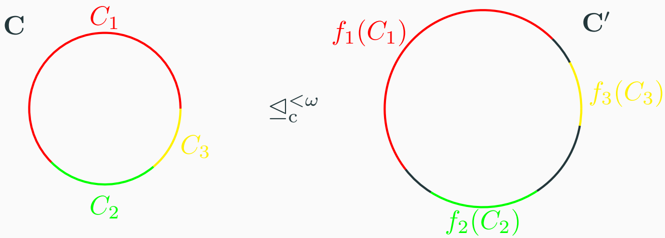
Definition

Let C and C_0 be circular orders. We say that C is a **convex** of C_0, and write $C \trianglelefteq_c C_0$, if there exists a convex subset A of C_0 such that $C \cong_{CO} A$. We denote by $(\trianglelefteq_c)_{CO}$ the restriction of the convexity relation to the set CO of (codes for) countable circular linear orders.

The convexity relation $\trianglelefteq_{\mathrm{c}}$ on CO is **not** transitive.



Definition

Let $C, C' \in CO$. Then $C \trianglelefteq_{\mathrm{c}}^{<\omega} C'$ if and only if there exists $k \in \omega$ and (non necessarily infinite) convex subsets $C_1, \ldots, C_k$ of C such that

- $C = C_1 + ... + C_k$, and
- for every $i = 1, ..., k$ there exists $f_i : C_i \to C'$ witnessing $C_i \trianglelefteq_{\mathrm{c}} C'$ such that the $f_i(C_i)$'s are pairwise disjoint in C' and

$$C'(f_i(x_i), f_j(y_j), f_h(z_h))$$

for every $x_i \in C_i, y_j \in C_j, z_h \in C_h$ and $i < j < h \leq k$.

$(\trianglelefteq_{\mathrm{c}}^{<\omega})_{CO}$ is an analytic quasi-order on CO.

Denote by $(\bowtie_c^{<\omega})_{CO}$ its induced (analytic) equivalence relation.

Theorem [I. - Marcone - Motto Ros]

$\cong_{LO} \leq_B \bowtie_c^{<\omega}$.

Knots and their classification

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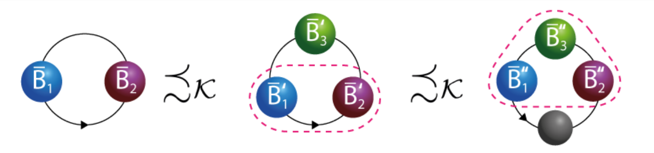
A component notion for Knots?

One may be tempted to transfer the component relation from proper arcs to knots through the transformation $(\bar{B}, f) \mapsto K_{(\bar{B},f)}$.

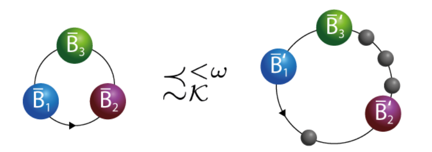
> *Given two knots $K, K' \in \mathcal{K}$, we say that K is a* ***component*** *of K', and write $K \precsim_{\mathcal{K}} K'$, if $K \equiv_{\mathcal{K}} K_{(\bar{B}', K' \cap \bar{B}')}$ for some sub-arc $(\bar{B}', K' \cap \bar{B}')$ of K'.*

However, the choice of the cutting point for a knot is in general not unique and one may produce unexpected situations. Moreover, $\precsim_{\mathcal{K}}$ is **not transitive**.



Then we consider the "transitivization" of the $\precsim_{\mathcal{K}}$ and introduce the (finite) piecewise component relation $\precsim_{\mathcal{K}}^{<\omega}$.

The (finite) piecewise mutual component relation



Definition

Let $K, K' \in \mathcal{K}$. Then K is a **(finite) piecewise component** of K', in symbols

$$K \precsim^{<\omega}_{\mathcal{K}} K',$$

if and only if there is an orientation of K' and a finite number of closed balls $\bar{B}'_1, ..., \bar{B}'_n$ such that

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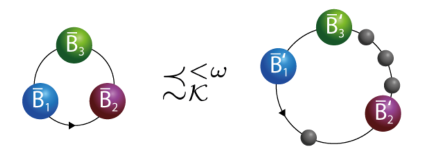

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$$K \precsim_{\mathcal{K}}^{<\omega} K',$$

if and only if there is an orientation of K' and a finite number of closed balls $\bar{B}'_1, ..., \bar{B}'_n$ such that

(a) the $(\bar{B}'_i, K' \cap \bar{B}'_i)$ are (almost) pairwise disjoint sub-arcs of K', oriented according to the chosen orientation of K', of which K is an "ordered" **(finite) tame sum**;

(b) if an endpoint of some $(\bar{B}'_i, K' \cap \bar{B}'_i)$ is singular, then it is not isolated.

The relation $\precsim_{\mathcal{K}}^{<\omega}$ is an analytic quasi-order on $\mathcal{K}$.

Denote by $\approx_{\mathcal{K}}^{<\omega}$ its associated (analytic) equivalence relation and call it the **(finite) piecewise mutual component relation**.

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Proposition

Let $K \in \mathcal{K}$, and let $I_{\mathcal{K}}$ be the trivial knot, i.e. a fixed representative of the $\equiv_{\mathcal{K}}$-class of circles embedded in S^3. Then the following are equivalent:

1) K is tame ; 2) $K \approx_{\mathcal{K}}^{<\omega} I_{\mathcal{K}}$; 3) $K \precsim_{\mathcal{K}}^{<\omega} I_{\mathcal{K}}$.

In particular, the $\approx_{\mathcal{K}}^{<\omega}$-class of the trivial/tame knots is a minimum for (the quotient order of) $\precsim_{\mathcal{K}}^{<\omega}$.

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Theorem [I. - Marcone - Motto Ros]

- $(\trianglelefteq_c^{<\omega})_{CO} \leq_B \precsim_{\mathcal{K}}^{<\omega}$. Then, we have $(\bowtie_c^{<\omega})_{CO} \leq_B \approx_{\mathcal{K}}^{<\omega}$.
- $\cong_{CO} \sim_B \cong_{LO} \leq_B \approx_{\mathcal{K}}^{<\omega}$.

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Thank you for your attention!