OCA in the class of hereditarily Lindelöf submetrizable spaces

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Open partitions/colorings

Given a topological space X and $K \subseteq [X]^2$ we say that K is open if the set

$$\{\langle x, y\rangle \colon \{x, y\} \in K\}$$

is open in X^2. A partition $[X]^2 = K_0 \sqcup K_1$ so that K_0 is open is called an open partition. Given a topological space X, consider the following "Open coloring Axiom/dichotomy" about partitions of $[X]^2$.

Definition (Todorčević '89)

- OCA(X): For every open partition $[X]^2 = K_0 \sqcup K_1$ either
  1. There exists an uncountable 0-homogeneous set $Y \subseteq X$ (*i.e.*, $[Y]^2 \subseteq K_0$), or else
  2. X is the union of countably many 1-homogeneous sets (*i.e.*, $X = \bigcup_{n\in\omega} X_n$ with $[X_n]^2 \subseteq K_1$ for $n \in \omega$).
- OCA*(X): For every open partition $[X]^2 = K_0 \sqcup K_1$ either
  1. There is a perfect 0-homogeneous set $P \subseteq X$ (*i.e.*, P is a nonempty closed subset without isolated points), or else
  2. X is the union of countably many 1-homogeneous sets.

It is well know that $\mathsf{OCA}^*(X)$ is considered as a two-dimensional version of $\mathsf{PSP}(X)$. Thus, $\mathsf{OCA}^*(X)$ is more relevant to the study of definable sets of X.

Definition (Todorčević '89)

- $\mathsf{OCA}(X)$: For every open partition $[X]^2 = K_0 \sqcup K_1$ either
  1. There exists an uncountable 0-homogeneous set $Y \subseteq X$, or else
  2. X is the union of countably many 1-homogeneous sets.
- $\mathsf{OCA}^*(X)$: For every open partition $[X]^2 = K_0 \sqcup K_1$ either
  1. There is a perfect 0-homogeneous set $P \subseteq X$, or else
  2. X is the union of countably many 1-homogeneous sets.

Our main motivation comes from the following general problem:

Problem (Todorčević '89)

1. *For which topological spaces X the statement $\mathsf{OCA}(X)$ is true in ZFC (or relatively consistent with ZFC)?*
2. *For which topological spaces X the statement $\mathsf{OCA}^*(X)$ is true in ZFC?*

This is still a widely open problem.

Known results from the literature

Problem (Todorčević '89)

1. *For which topological spaces X the statement* $\mathrm{OCA}(X)$ *is true in* ZFC *(or relatively consistent with* ZFC*)?*
2. *For which topological spaces X the statement* $\mathrm{OCA}^*(X)$ *is true in* ZFC*?*

- $\mathrm{OCA}(X)$ implies that the set of isolated points of X is at most countable.
- Let X and Y be topological spaces so that Y is Hausdorff and there is a continuous surjection function $f\colon X \to Y$. Then $\mathrm{OCA}(X)$ (resp., $\mathrm{OCA}^*(X)$) implies $\mathrm{OCA}(Y)$ (resp., $\mathrm{OCA}^*(Y)$).
- $\mathrm{OCA}^*(\omega^\omega)$ is true and hence OCA^* holds for analytic sets.
- Every Hausdorff second countable space is a 1-1 continuous image of a set of reals (*i.e.*, a subset of 2^ω or ω^ω). Therefore, $\mathrm{OCA}(X)$ for sets of reals X implies $\mathrm{OCA}(Y)$ for Hausdorff second countable spaces Y.
- (Todorčević '89) PFA implies OCA (*i.e.*, $\mathrm{OCA}(X)$ holds for every separable metrizable space X).

Outside of the class of separable metrizable spaces, open colorings on topological spaces has been little studied. Since OCA has a strong influence on structures closely related to separable metric spaces, exploring OCA in a wider class could have interesting applications.

Conjecture (Todorčević '89)

It is relatively consistent with ZFC *that if* X *is a regular topological space with no uncountable discrete subspaces then* $\mathrm{OCA}(X)$ *holds.*

This conjecture asks to weaken the topological assumptions on the space X to essentially the best possible.

- (Todorčević '89) (PFA) Every regular topological space either contains an uncountable discrete subspace or else is hereditarily Lindelöf (HL).

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Conjecture (Todorčević '89)

It is relatively consistent with ZFC *that if X is a regular HL topological space then* OCA(X) *holds.*

A careful inspection of the proof of $\mathrm{OCA}^*(\omega^\omega)$ provides us the following generalization.

Theorem

Let X be a HL strong Choquet submetrizable space. Then $\mathrm{OCA}^*(X)$ *holds.*

- A topological space X is submetrizable if X has a weaker metrizable topology.
- Given a nonempty topological space X, the strong Choquet game G^s_X is a two-player game defined as follows: Players I and II take turns to play open sets U_i, V_i and elements $x_i \in X$, $i \in \omega$. The rules of the game are $x_i \in V_i \subseteq U_i$ and $U_{i+1} \subseteq V_i$, for all $i \in \omega$. Player II wins the play if $\bigcap_i U_i = \bigcap_i V_i \neq \emptyset$.
- A topological space X is a strong Choquet space if Player II has a winning strategy in the strong Choquet game G^s_X.

Sketch of the proof

Fix a metric ρ on X where every ρ-open ball is open in X and fix an open partition $[X]^2 = K_0 \sqcup K_1$ such that X cannot be covered by countably many 1-homogeneous sets. Let σ be a winining strategy for player II in the strong Choquet game G_X^s. Recursively on $s \in 2^{<\omega}$ construct a family $\langle U_s : s \in 2^{<\omega}\rangle$ so that:

1. $U_s \subseteq X$ is an open set for all $s \in 2^{<\omega}$;
2. $U_s \supseteq U_t$ if $t \supseteq s$;
3. if $s \in 2^{<\omega}$, then $\overline{U_{s^\frown 0}}^{\rho} \cap \overline{U_{s^\frown 1}}^{\rho} = \emptyset$;
4. $U_{s^\frown 0} \otimes U_{s^\frown 1} \subseteq K_0$; (For $A, B \subseteq X$, $A \otimes B$ denotes the set $\{\{x, y\} \in [X]^2 \colon \langle x, y\rangle \in A \times B$ or $\langle y, x\rangle \in A \times B\}$)
5. U_s cannot be covered by countably many 1-homogeneous sets for all $s \in 2^{<\omega}$; and
6. $diam_\rho(U_s) < 1/2^{|s|}$ for all $s \in 2^{<\omega}$.

To guarantee these conditions, the recursion step needs the following claim:

Claim

Let U be an open set such that can not be covered by countably many 1-homogeneous set. Then there exist $x_0 \neq x_1 \in U$ and two disjoint sets $U^0, U^1 \subseteq U$ so that:

1. $x_i \in U^i$ for $i \in 2$;
2. $U^0 \otimes U^1 \subseteq K_0$; and
3. U^i can not be covered by countably many 1-homogeneous sets for all $i \in 2$.

Proof.

Define $\mathcal{F}$ as the family of all open sets $V \subseteq U$ that can be covered by countably many 1-homogeneous sets. Since X is hereditarily Lindelöf, there exsist a countable family $\mathcal{F}' \subseteq \mathcal{F}$ with $\bigcup \mathcal{F}' = \mathcal{F}$. Then $U \setminus \bigcup \mathcal{F}'$ cannot be the empty set; furthermore, it cannot be covered by countable many 1-homogeneous sets. Let $x_0 \neq x_1 \in U \setminus \bigcup \mathcal{F}'$ be such that $\{x_0, x_1\} \in K_0$ and let $U^0, U^1 \subseteq U$ open neighborhood of x_0, x_1 respectively. Without loss of generality $U^0 \cap U^1 = \emptyset$.Thus, by the choice of x_i, U^i can not be covered by countably many 1-homogeneous sets for all $i \in 2$. □

By recursion on s we can construct U_s. Let $U_\emptyset = X$. Using the previous claim we can find $x_0 \neq x_1 \in U_\emptyset$ and disjoint sets $U^0, U^1 \subseteq U_\emptyset$ with the properties from the claim. Without loss of genreralitiy, assume that $diam_\rho(U^i) < 1/2$ and $\overline{U^0}^\rho \cap \overline{U^1}^\rho = \emptyset$. Let $U_{\langle i \rangle} = U^i$, $x_{\langle i \rangle} = x_i$, and $V_{\langle i \rangle} = \sigma(\langle x_{\langle i \rangle}, U_{\langle i \rangle} \rangle)$, then V_i is not a σ-1-homogeneous set. Assume x_s, U_s, and V_s has been defined. Similarly there exists $x_{s^\frown 0}, x_{s^\frown 1} \in V_s$ and disjoint sets $U^0, U^1 \subseteq V_s$ with the properties from the claim with $diam_\rho(U^i) < 1/2^{|s|}$. We put $U_{s^\frown i}$ as in the claim and $V_{s^\frown i} = \sigma(\langle x_{s \restriction 0}, U_{s \restriction 0} \rangle, V_{s \restriction 0}, \ldots, \langle x_s, U_s \rangle, V_s, \langle x_{s^\frown i}, U_{s^\frown i} \rangle)$.

Since X is strong Choquet, $\bigcap_{n \in \omega} U_{f \restriction n} = \bigcap_{n \in \omega} V_{f \restriction n} \neq \emptyset$ for every $f \in 2^\omega$. By construction, $diam_\rho(U_s) < 1/2^{|s|}$ for every $s \in 2^{<\omega}$ and hence $\bigcap_{n \in \omega} U_{f \restriction n} = \bigcap_{n \in \omega} \overline{U_{f \restriction n}}^\rho$ is only one point for every $f \in 2^\omega$. Therefore, the set

$$\bigcap_{n < \omega} \bigcup_{s \in 2^n} U_s = \bigcap_{n < \omega} \bigcup_{s \in 2^n} \overline{U_s},$$

forms a perfect 0-homogeneous set. □

Theorem

Let X be a HL strong Choquet submetrizable space. Then $\mathrm{OCA}^*(X)$ holds.

- (Choquet) A topological space is Polish iff it is T_1, regular, second countable, and strong Choquet.
- If we let $\mathcal{S}^*$ denote the class of all HL strong Choquet submetrizable spaces, then $\mathcal{S}^*$ forms a good class that extends the class of Polish spaces.
- Examples of non-Polish spaces that belong to $\mathcal{S}^*$ are: the Sorgenfrey line $\mathbb{S}$, the space ω^ω with the Gandy-Harrington topology τ_{GH}: the topology generated by the (lightface) Σ^1_1 subsets of the Baire space ω^ω.

The idea of the Gandy-Harrington topology was introduced in the 1960's by Gandy, but was not used too much until Harrington used it to give, in the 1970's, a much simpler proof of Silver's dichotomy:

Theorem (Silver, 1980)

Let X be a Polish space and E a co-analytic (as a subset of X^2) equivalence relation on X. Then either X/E is countable or else there is a Cantor set $C \subseteq X$ consisting of pairwise E-inequivalent elements.

Following the Harrington's argument and using $OCA^*(\omega^\omega, \tau_{GH})$ also it is possible obtain a proof of Silver's dichotomy.

Conjecture (Todorčević '89)

It is relatively consistent with ZFC *that if X is a regular HL topological space then* OCA(X) *holds.*

Question

Is it relatively consistent with ZFC that OCA(X) holds for every HL submetrizable space X?

We will denote by $\mathcal{S}$ to the class of all HL submetrizable spaces.

- Let X and Y be topological spaces so that Y is Hausdorff and there is a continuous surjection function $f\colon X \to Y$. Then OCA(X) (resp., OCA*(X)) implies OCA(Y) (resp., OCA*(Y)).

Searching a surjective basis for $\mathcal{S}^*$

- Let X and Y be topological spaces so that Y is Hausdorff and there is a continuous surjection function $f\colon X \to Y$. Then OCA(X) (resp., OCA*(X)) implies OCA(Y) (resp., OCA*(Y)).

Question

Is there a surjective basis for $\mathcal{S}^*$? *i.e.*, Is there a subclass $\mathcal{S}_0 \subseteq \mathcal{S}^*$ so that

1. for every $Y \in \mathcal{S}^*$ there exists an $X \in \mathcal{S}_0$ such that Y is a continuous image of X, and
2. ensuring that the choice of $\mathcal{S}_0$ is minimal (in the sense of cardinality)?

Question

Is each $Z \in \mathcal{S}$ a continuous image of a subspace Y of $X \in \mathcal{S}^*$?

Partial results

Theorem

Let (X, τ) be any element of $\mathcal{S}^*$ with τ_M a metric topology on X witness that τ is submetrizable. Then

1. $\tau \subseteq \Sigma^1_1(\tau_M)$.
2. $(X, \tau_M) \in \Sigma^1_1(\hat{X}, \hat{\tau}_M)$, *i.e.*, (X, τ_M) *is absolutely analytic in its metric completion.*
3. *There is a topology τ_X on ω^ω extending the usual topology on ω^ω so that $(\omega^\omega, \tau_X) \in \mathcal{S}^*$ and (X, τ) is a continuous image of (ω^ω, τ_X).*

Let $\mathcal{S}^*_B$ be the subclass of $\mathcal{S}^*$ consisting of all spaces of type (ω^ω, τ_X).

Question

Is there a surjective finite basis for $\mathcal{S}^*_B$?

- Clearly $\mathbb{S}, \omega^\omega, (\omega^\omega, \tau_{GH}) \in \mathcal{S}^*_B$.

Question

Is it relatively consistent with ZFC that $\mathrm{OCA}(X)$ holds for every $X \in \mathcal{S}$?

Question

Is each $Z \in \mathcal{S}$ a continuous image of a subspace Y of $X \in \mathcal{S}_B^*$?

Given $X \in \mathcal{S}^*$ we denoted by OCA_X the statement $\mathrm{OCA}(Y)$ holds for every subspace $Y \subseteq X$. Thus, $\mathrm{OCA}_{(\omega^\omega, \tau_{GH})} \equiv \mathrm{OCA}_{\omega^\omega} \equiv \mathrm{OCA}$.

Question

Is it relatively consistent with ZFC that OCA_X holds for every $X \in \mathcal{S}_B^*$?

- (Peng) $\mathrm{OCA}_{\mathbb{S}}$ is equivalent to OCA.

Question

Are OCA and OCA_X equivalent for every $X \in \mathcal{S}_B^*$?

Thank you for your attention!